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HÖLDER SPECTRUM OF FUNCTIONS MONOTONE IN SEVERAL VARIABLES

This talk was about the results of a joint paper with Stéphane Seuret. Details and proofs of the results mentioned in this abstract are in [5].

Let d be an integer greater than one. A function $f : [0,1]^d \to \mathbb{R}$ is continuous monotone increasing in several variables (in short: MISV) if for all $i \in \{1, ..., d\}$, the functions $f^{(i)}(t) = f(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d)$ are continuous monotone increasing. We use the notation $\mathcal{M}^d = \{f \in C([0,1]^d) : f \text{ MISV}\}$. The space $\mathcal{M}^d$ is a separable complete metric space when equipped with the supremum, L^∞ norm.

The multifractal properties of functions in $\mathcal{M}^1$ were studied in [3]. In this paper, we deal with the higher dimensional case. This is also a continuation of [4] where multifractal properties of typical/generic Borel measures on $[0,1]^d$ were investigated.

Definition 1. Let $f \in L^\infty([0,1]^d)$. For $h \geq 0$ and $x \in [0,1]^d$, the function f belongs to C_x^h if there are a polynomial P of degree less than $[h]$ and a constant C such that, for x' close to x, $|f(x') - P(x'-x)| \leq C|x'-x|^h$. The pointwise Hölder exponent of f at x is $h_f(x) = \sup\{h \geq 0 : \ f \in C_x^h\}$.

The singularity spectrum of f is defined by $d_f(h) = \dim_H E_f^h$, where $E_f^h = \{x : h_f(x) = h\}$.

We will also use the sets $E_f^{h,\leq} = \{x : h_f(x) \leq h\} \supset E_f^h$.

Our main results are the following theorems.

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Theorem 2. *For all $f \in \mathcal{M}^d$ and $h \geq 0$, we have*

$$\dim_H E_f^{h,\leq} \leq \min(d-1+h, d). \tag{1}$$

In particular, $d_f(h) = \dim_H(E_f^h) \leq \min(d-1+h, d)$.

Theorem 3. *There exists a dense G_δ set $\mathcal{R} \subset \mathcal{M}^d$ such that for all $f \in \mathcal{R}$ we have $d_f(h) = d-1+h$ for all $h \in [0,1]$. For these functions, for every $h > 1$ the set E_f^h is empty.*

In the second part of our paper we study level sets of MISV functions. We define for every $a \in \mathbb{R}$ the level set $L_f(a)$ by $L_f(a) = \{x \in [0,1]^d : f(x) = a\}$.

We prove the following.

Theorem 4. *There exists a dense G_δ subset $\mathcal{L}$ in $\mathcal{M}^d$ such that for all $f \in \mathcal{L}$ the following holds.*

There exist a set $X_f \subset [0,1]^d$ and a set $A_f \subset (f(0,...,0), f(1,...,1)) = (m_f, M_f)$ satisfying:

(i) $\dim_H X_f = d-1$, $\dim_H A_f = 0$,

(ii) for every $a \in (m_f, M_f)$, there is at most one point of $L_f(a)$ which does not belong to X_f (in other words, $L_f(a) \cap ([0,1]^d \setminus X_f)$ contains at most one point).

(iii) for every $a \in (m_f, M_f) \setminus A_f$, $L_f(a) \subset X_f$.

The level sets of generic continuous MISV functions are quite simple compared to the level sets of generic continuous functions (see for example [2]).

Set $\mathbb{R}^d_+ = \{(l_1, ..., l_d) : \forall i,\ l_i \geq 0\}$ and $\mathbb{R}^d_- = \{(l_1, ..., l_d) : \forall i,\ l_i \leq 0\}$. It is well-known that generic continuous functions on $[0,1]$ are nowhere monotone (see for example [1], Chapter 10). MISV functions are obviously monotone increasing along lines $\underline{l}t + \underline{b} = (l_1 t + b_1, ..., l_d t + b_d)$, $(t \in \mathbb{R})$ if $\underline{l} \in \mathbb{R}^d_+$ and monotone decreasing if $\underline{l} \in \mathbb{R}^d_-$. For the generic functions in $\mathcal{M}^d$ one cannot say much more:

Theorem 5. *There exists a dense G_δ subset $\mathcal{G}$ in $\mathcal{M}^d$ such that for any $f \in \mathcal{G}$, if $\underline{l} = (l_1, ..., l_d) \notin \mathbb{R}^d_+ \cup \mathbb{R}^d_-$ and $\underline{b} = (b_1, ..., b_d) \in \mathbb{R}^d$, then the function $g_{\underline{l},\underline{b}}(t) = f(\underline{l}t + \underline{b})$, $t \in \mathbb{R}$ is monotone on no non-empty open subinterval on its domain.*

References

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