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ON SOME PROPERTIES OF VAN KOCH CURVES

Our research goes back to the so-called problem of AC-removability of quasiconformal curves. The solution to that problem was furnished making use of Van Koch curves which turned out to be quasiconformal but not AC-removable.

The result was based on the investigation of integrals of type

$$T(f)(z) = \int_{\Gamma} \frac{f(\zeta)d\mu(\zeta)}{\zeta - z}, \quad z \in \mathbb{C} \tag{1}$$

where $\Gamma \subset \mathbb{C}$ is a Van Koch curve, μ a finite measure on Γ, and $f : \Gamma \to \mathbb{C}$ is essentially bounded.

Definition 1. *Let Ω be an open subset of $\mathbb{C}$. A set $E \subset \Omega$, closed in Ω, is called AC-removable in Ω if each continuous function $f : \Omega \to \mathbb{C}$ analytic in $\Omega \setminus E$, is also analytic in Ω. In our context $\Omega = \mathbb{C}$.*

A curve $C \subset \mathbb{C}$ is said to be quasiconformal if it is a quasiconformal image of a closed interval.

1. A special family of Van Koch's curves

We consider the family of Van Koch curves (see [1] for details and further references).

$$\{\Gamma_\theta : \theta \in (0, \pi/4)\} \tag{2}$$

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obtained from the triangle Δ_1^0 with vertices $0, 1, (1 + i\tan\theta)/2$, consecutively deleting some special sequence of isosceles open triangles. In the first step we delete from Δ_1^0 (the triangle of rank zero) the open triangle[1] with vertices $\lambda^2, 1 - \lambda^2, (1 + i\tan\theta)/2$, where $\lambda = (2\cos\theta)^{-1}$. We get two closed triangles Δ_1^1, Δ_2^1 of rank one, each one being similar to Δ_1^0. In the n^{th} step, we obtain 2^n equal triangles Δ_k^n, of rank n, similar to Δ_1^0, $\mathrm{diam}\Delta_k^n = \lambda^n$. By definition,

$$\Gamma = \Gamma_\theta = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^n} \Delta_k^n. \tag{3}$$

2. The natural parametrization of Van Koch's curves

For each fixed $\theta \in (0, \pi/4)$ there exist two sequences $\{L^n\}$, $\{\varphi_n\}$, corresponding to the representation (3). Elements of the first sequence are polygonal arcs

$$L^n = \bigcup_{k=1}^{2^n} s_k^n, \tag{4}$$

where $s_k^n = [z_{k-1}^n, z_k^n]$ is the line segment with endpoints z_{k-1}^n, z_k^n ($z_0^n = 0$, $z_{2^n}^n = 1$) which is the side of Δ_k^n lying opposite the angle $\pi - 2\theta$. Each s_k^n is oriented from z_{k-1}^n to z_k^n. We let L^0 denote the segment $[0, 1]$, so we have $L^0 = s_1^0 = [0, 1]$.

The second sequence consists of the homeomorphisms $\varphi_n : [0, 1] \to L^n$ with the properties:
(i) $\varphi_n(0) = 0, \varphi_n(1) = 1$;
(ii) for each k, $1 \le k \le 2^n$, the restriction $\varphi_n | I_k^n$, where $I_k^n = [(k-1)2^{-n}, k2^{-n}]$, is an affine mapping of I_k^n onto s_k^n, $\varphi_n(k2^{-n}) = z_k^n$ for $k \in \{0, 1, \ldots, 2^n\}$. Thus φ_n is a parametrization of the oriented arc L^n.

Theorem 1. *(i) For each $\theta \in (0, \pi/4)$ there exists a uniform limit $\varphi = \lim \varphi_n$ which is a homeomorphism of $[0, 1]$ onto Γ_θ.*

(ii) The homeomorphism $\varphi : [0, 1] \to \Gamma_\theta$ satisfies bilateral Hölder's inequality

$$A|t_1 - t_2|^{\log_2 2\cos\theta} \le |\varphi(t_1) - \varphi(t_2)| \le 4|t_1 - t_2|^{\log_2 2\cos\theta} \tag{5}$$

where $A = A(\theta)$.

[1] Correction: in [1], Section 1, right after formula (3), the words: "similar to the initial triangle Δ_1^0" were written accidentally and, of course, should be deleted.

We call $\varphi(=\varphi_\theta)$ the natural parametrization of $\Gamma = \Gamma_\theta$ (for the sake of simplicity, we often omit θ).

The bilateral inequality (5) of Theorem 1 implies that Γ satisfies the so-called Ahlfors condition (see [2], [3]) which is important in the proof of

Theorem 2. *(a) Each* $\Gamma = \Gamma_\theta,\ 0 < \theta < \pi/4,$ *is a quasiconformal curve.*

(b) The Hausdorff dimension of Γ_θ *equals* $1/log_2 2cos\theta$

(c) None of $\Gamma_\theta,\ 0 < \theta < \pi/4,$ *is AC-removable.*

3. A Continuous Cauchy-type Integral

We prove the existence of the integral (1) in the Lebesgue sense where μ is the image of Lebesgue measure from $[0, 1]$ onto Γ via φ.

We have, in particular, $\mu\Gamma_k^n = 2^{-n}$ for each $n \in \{0\} \cup \mathbb{N}$ and $k = 1, \ldots, 2^n$, where $\Gamma_k^n = \Gamma \cap \Delta_k^n$. For $n = 0$ we have $\Gamma_1^0 = \Gamma \cap \Delta_1^0 = \Gamma$ and $\mu\Gamma = 1$.

In what follows, we consider the space $L^\infty(\Gamma) = L^\infty(\Gamma, \mu)$ of μ-measurable essentially bounded functions $f : \Gamma \to \overline{\mathbb{C}}$. We equip $L^\infty(\Gamma)$ with its natural essential sup-norm $||f||_\infty$.

Given any $f \in L^\infty(\Gamma)$, we let for each $z \in \mathbb{C}$

$$T(f)(z) = \int_\Gamma \frac{f(\zeta)d\mu(\zeta)}{\zeta - z}. \tag{6}$$

Theorem 3. *For each* $f \in L^\infty(\Gamma)$ *the function* $\mathbb{C} \ni z \mapsto T(f)(z)$ *is analytic in* $\mathbb{C} \setminus \Gamma$, *continuous in* $\mathbb{C}$ *and vanishing at* ∞.

4. Linear operator $T : L^\infty(\Gamma) \to \mathcal{AC}(\Gamma)$

By $\mathcal{AC}(\Gamma)$ we denote the space of functions $f : \mathbb{C} \to \mathbb{C}$ continuous in $\mathbb{C}$, analytic in $\mathbb{C} \setminus \Gamma$ and vanishing at ∞. We equip $\mathcal{AC}(\Gamma)$ with the usual sup norm.

The mapping $T : L^\infty(\Gamma) \to \mathcal{AC}(\Gamma)$ defined by (6) is a well-posed linear operator.

Theorem 4. *(i)* $kerT = \{0\}$.
(ii) T is a compact operator.

5. Logarithmic potential defined for Van Koch curves

Given an $f \in L^\infty(\Gamma)$, we let for each $z \in \mathbb{C}$

$$P(f)(z) = \int_\Gamma f(\zeta) \ln|\zeta - z| d\mu(\zeta). \tag{7}$$

and call $P(f)$ a logarithmic potential with density f.

Theorem 5. *For each $f \in L^\infty(\Gamma)$ the partial derivatives $P_x = \frac{\partial P}{\partial x}$, $P_y = \frac{\partial P}{\partial y}$ are continuous in $\mathbb{C}$ [4].*

References

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