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ON THE LEVEL SETS OF THE TAKAGI FUNCTION

1 Introduction.

Takagi's continuous nowhere differentiable function is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \phi(2^n x), \tag{1}$$

where $\phi(x) = \operatorname{dist}(x, \mathbb{Z})$, the distance from x to the nearest integer. Since its initial discovery in 1903 by Takagi [12], many properties of the Takagi function and various of its generalizations have been investigated, but the function itself has been slow to give up some of its deepest secrets. For example, it has only recently been established at which set of points $T(x)$ has an infinite derivative (Allaart and Kawamura [3], Krüppel [8]). There has also been a great deal of interest in recent years in the level set structure of T, and that is the focus of this summary.

2 Overview of results.

For $y \in [0, \frac{2}{3}]$, define

$$L(y) = \{x \in [0,1] : T(x) = y\}.$$

Thus, $L(y)$ is the level set at level y of the Takagi function. Since $T(x) > 0$ for all $x \in (0,1)$, the simplest level set is $L(0) = \{0, 1\}$. At the other extreme,

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Kahane [6] showed that $L(\frac{2}{3})$ is the set of all $x \in [0,1]$ whose binary expansion $x = 0.b_1b_2b_3\ldots$ satisfies $b_{2i-1} + b_i = 1$ for all $i \in \mathbb{N}$. As a result, $L(\frac{2}{3})$ is a Cantor set of Hausdorff dimension $\frac{1}{2}$. Surprisingly, a more general study of the level sets of T was apparently not undertaken until 2005, when Knuth [7, p. 103] published an algorithm for determining $L(y)$ for rational y. (It is however not known whether this algorithm always halts.) A few years later, Buczolich [5] showed the following:

Theorem 1 (Buczolich, 2008)**.** *For almost every y (with respect to Lebesgue measure λ on $[0, \frac{2}{3}]$), $L(y)$ is a finite set.*

This may come as a bit of a surprise. The Takagi function is self-affine, and self-affine functions ordinarily have level sets which are uncountable (indeed, of positive Hausdorff dimension) almost everywhere. But the graph of T is only a borderline fractal: it has Hausdorff dimension 1. This may help explain why it behaves unlike other fractal functions in several other respects as well.

Although almost every level set of T is finite, the level sets can be quite large. In fact, they are infinitely large on average, as shown by Lagarias and Maddock [10]; see also [1].

Theorem 2 (Lagarias and Maddock, 2010)**.** *The expected cardinality of a level set $L(y)$ for y chosen at random from $[0, \frac{2}{3}]$ is infinite. That is,*

$$\mathrm{E}|L(y)| = \frac{3}{2}\int_0^{2/3} |L(y)|\,dy = \infty.$$

Another way of viewing the level sets is through the lens of Baire category. This gives a marked contrast with Theorem 1.

Theorem 3 (Allaart, 2011 [1])**.** *Let*

$$S_\infty^u = \{y : L(y) \text{ is uncountable}\}.$$

Then $S_\infty^u = E \cup M$, where E is a dense G_δ, and M is a countable set consisting of the local maximum values of T. In particular, S_∞^u is residual in $[0, 2/3]$.

2.1 Local level sets

Lagarias and Maddock [9] partition the level sets of T into easier to understand pieces which they call *local level sets*. Their definition requires the following notation. For $x \in [0,1)$, write x in binary as

$$x = \sum_{k=1}^{\infty} 2^{-k}\varepsilon_k, \qquad \varepsilon_k \in \{0,1\}.$$

(If x is dyadic rational, we take the expression with $\varepsilon_k = 0$ eventually.) Define

$$D_n(x) = \sum_{k=1}^{n} (-1)^{\varepsilon_k}, \qquad n = 0, 1, 2, \dots$$

Define an equivalence relation $\sim$ on $[0, 1)$ by

$$x \sim x' \quad \Longleftrightarrow \quad |D_n(x)| = |D_n(x')| \ \ \forall n.$$

The *local level set* of $x \in [0, 1)$ is the set

$$L_{loc}(x) := \{x' \in [0, 1) : x' \sim x\}.$$

(Lagarias and Maddock defined local level sets slightly differently, but the difference has no bearing on the results presented here.)

The pertinent fact here is that if $x \sim x'$, then $T(x) = T(x')$, so local level sets are subsets of (global) level sets. Furthermore, each local level set is either finite (with cardinality 2^k for some $k \in \mathbb{N}$) or a Cantor set. It turns out that measure theory and Baire category disagree also on whether the typical level set contains just a few or infinitely many local level sets.

Theorem 4 (Lagarias and Maddock, 2010 [9])**.** *The average number of local level sets in a level set is* $3/2$. *That is,*

$$\mathrm{E}[N^{loc}(y)] := \frac{3}{2} \int_0^{2/3} N^{loc}(y)\, dy = \frac{3}{2},$$

where $N^{loc}(y)$ *denotes the number of local level sets contained in* $L(y)$.

On the other hand, we have the following:

Theorem 5 (Allaart, 2011 [1])**.** *The set*

$$\{y : L(y) \text{ contains infinitely many local level sets}\}$$

is residual in $[0, \frac{2}{3}]$, *and the set*

$$\{y : L(y) \text{ contains uncountably many local level sets}\}$$

is dense in $[0, \frac{2}{3}]$, *and intersects each subinterval of* $[0, \frac{2}{3}]$ *in a continuum.*

2.2 Hausdorff dimension

As it is by now clear that, Buczolich's theorem notwithstanding, there are many level sets which are countably infinite, it is natural to ask "how many" level sets have in fact positive Hausdorff dimension, and how large the Hausdorff dimension of a level set can be. The first result in this direction was given by Maddock [11].

Theorem 6 (Maddock, 2010). *The intersection of the graph of T with any line of integer slope has Hausdorff dimension at most* 0.668. *In particular,* $\dim_H L(y) \le 0.668$ *for all* y.

Maddock conjectured that the maximal Hausdorff dimension is in fact $\frac{1}{2}$. That this is so was proved very recently by de Amo et al. [4].

Theorem 7 (de Amo et al., 2011). *The Hausdorff (and box-counting) dimension of $L(y)$ is at most $\frac{1}{2}$ for every y.*

Another interesting result indicating the prevalence of level sets with positive Hausdorff dimension is the following.

Theorem 8 (Lagarias and Maddock, 2010 [10]). *The set*

$$\{y \in [0, 2/3] : \dim_H L(y) > 0\}$$

has Hausdorff dimension 1.

2.3 Finite cardinalities

In this section we focus on the cardinalities of the finite level sets. Which cardinalities are possible, and do they occur with positive probability if a level y is chosen at random?

The results in this subsection are proved in [2]. Let

$$S_n = \{y : |L(y)| = n\}, \qquad n \in \mathbb{N}.$$

Not surprisingly perhaps, the most common finite cardinality is two.

Theorem 9 (Allaart, 2011). *Let λ denote Lebesgue measure. Then*

$$5/12 < \lambda(S_2) < 35/72,$$

so between 62.5% *and* 72.9% *of the level sets have cardinality 2.*

Theorem 10 (Allaart, 2011). *Every even positive integer is the cardinality of uncountably many level sets of T.*

More strongly, one could ask if each even cardinality occurs in fact with positive probability.

Conjecture 11. *For each $n \in \mathbb{N}$, $\lambda(S_{2n} > 0)$.*

The author can prove the conjecture for the following special cases:

Theorem 12 (Allaart, 2011)**.** *The Conjecture holds when n is*
(i) a power of 2, or
(ii) the sum or difference of two powers of 2.

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