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ON ASYMPTOTIC UNIFORM UPPER DENSITY IN LOCALLY COMPACT ABELIAN GROUPS

The notion of uniform asymptotic upper density – *a.u.u.d. for short* – on $\mathbb{R}$ goes back to the PhD thesis [6] of J.-P- Kahane in 1954, see [5] and [8], and was used first in Fourier analysis, but later on in many related areas. However, until recently no extension of the notion was known for locally compact abelian groups (*lca groups*) though they form a rather natural framework for many of the questions treated by means of this density.

The original construction of Kahane for $\mathbb{R}$ runs as follows. Let $S \subset \mathbb{R}$ be a sequence. If S is just a bounded perturbation of an arithmetic progression, that is writing $S = (s_k)_{k=-\infty}^{\infty}$ in increasing order $(\dots < s_k < s_{k+1} < \dots$ we have $s_k - L \cdot k = O(1)$ $(k \in \mathbb{Z})$, then we say that S has *uniform density* $D(S) := 1/L$. Now for a general sequence we can define

Defintion 1. *If $S \subset \mathbb{R}$ is a sequence, then*

$$\overline{D}^{\#}(S) := \inf\{D \ : \ \exists S' \supset S, D(S') = D\}. \tag{1}$$

Then Kahane shows that in fact this notion of density can be equivalently defined as follows.

Defintion 2. *If $S \subset \mathbb{R}$ is a sequence, then*

$$\overline{D}^{\#}(S) := \limsup_{r\to\infty} \sup_{x\in\mathbb{R}} \frac{\#\{s \in S \ : \ |s-x| \le r\}}{2r}. \tag{2}$$

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Actually, the lim sup is a limit, for the quantity "essentially decreases". Some people (e.g. Fürstenberg) call it Banach density.

A typical area of application is related to investigation of differences: for if $\overline{D}(S) > 0$, then $S - S$ has positive asymptotic density $\delta(S - S) \geq \overline{D}(S)$.

In $\mathbb{R}^d$ (or $\mathbb{Z}^d$) one can analogously consider, with a fixed basic set $K \subset \mathbb{R}^d$ like e.g. the unit ball or unit cube,

$$\overline{D}_K^{\#}(S) := \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d} \#(S \cap (rK + x))}{|rK|} \ .$$

Also, non-discrete, but locally Lebesgue-measurable sets arise in the context (in problems of plane geometry e.g.), where the natural density is defined by means of volume, not cardinality.

So let $K \subset \mathbb{R}^d$ be e.g. any *fat body.* Then a.u.u.d of a Lebesgue-measurable set $A \subset \mathbb{R}^d$ is defined as

$$\overline{D}_K(A) := \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d} |A \cap (rK + x)|}{|rK|} \ . \tag{3}$$

Clearly, the notion is translation invariant.

It is also well-known, that $\overline{D}_K(A)$ gives the same value for all nice bodies $K \subset \mathbb{R}^d$ (although this fact does not seem immediate from the formulation). To prove $\overline{D}_{K_1}(A) = \overline{D}_{K_2}(A)$ directly would require some tedious ε-covering of the boundary of K_1 by homothetic copies of K_2 etc. But we obtain this as a side result, being an immediate corollary of our Theorem 5, see Remark 6. Moreover, this way it follows elegantly for arbitrary measurable sets K, too.

The various ways we must encounter in measuring the size of S or A say in some translated and dilated copy of $K \subset \mathbb{R}^d$ motivates our further extension: we are aiming at asymptotic uniform upper densities of *measures*, say measure ν with respect to measure μ, and not only sets S or $A \subset \mathbb{R}^d$ (whether ν is related to μ (e.g. being the trace of μ on a set) or not).

The general formulation in $\mathbb{R}^d$ (or $\mathbb{Z}^d$) is thus

$$\overline{D}_K(\nu) := \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d} \nu(rK + x)}{|rK|} \ . \tag{4}$$

E.g. in (2) $\nu := \#$ is the cardinality or counting measure of a set S, while $\mu := |\cdot|$ is just the volume. However, to keep the meaningful properties of the original translation-invariant a.u.u.d., for μ the *Lebesgue measure (volume) remains the only reasonable choice.* This is so because of the *uniqueness of Haar measure* on groups.

Our heuristics in finding the key definitions for lca groups arose from the idea of grasping the fact that the set, where we may analyze relative densities of the given set A or measure ν, must grow large (as in case of $\mathbb{R}$ the dilated copies rK do). Then we encountered the following nice result in Rudin's book [13], see 2.6.7. Theorem on [13, p. 52].

Theorem 3. *If $\varepsilon > 0$ and $C \Subset G$, then there exists a Borel set V in G with compact closure, such that $\mu(V + C) < (1 + \varepsilon)\mu(V)$.*

Thinking of $\mathbb{R}^d$, it is natural to visualize the content of this lemma as for any given compact set C the difference between V and $V+C$ is just a bounded (compact) perturbation on the boundary of V, so if V is chosen quite large, than the change of volume becomes relatively negligible. This suggested us the idea of replacing limits and size restrictions by the simple trick of division by $\mu(V + C)$, in place of simply $\mu(V)$, in the definition of a.u.u.d., thus leading to (5). Indeed, if $\mu(V)$, that is V, is large enough – in the sense of the above Theorem 3 – then the increase of $\mu(V)$ to $\mu(V + C)$ does not matter asymptotically; and if V is not enough large, than the division by a larger measure set makes the corresponding quantity out of interest in the search of high relative density. That was our heuristical idea in the construction of Definition 4.

Defintion 4. *Let G be a LCA group and $\mu := \mu_G$ be its Haar measure. If ν is another measure on G with the sigma algebra of measurable sets being $\mathcal{S}$, then we define*

$$\overline{D}(\nu) := \overline{D}(\nu; \mu) := \inf_{C \Subset G} \sup_{V \in \mathcal{S} \cap \mathcal{B}_0} \frac{\nu(V)}{\mu(C + V)} \ . \tag{5}$$

In particular, if $A \subset G$ is Borel measurable and $\nu = \mu_A$ is the trace of the Haar measure on the set A, then we get

$$\overline{D}(A) := \overline{D}(\mu_A) := \overline{D}(\mu_A; \mu) := \inf_{C \Subset G} \sup_{V \in \mathcal{B}_0} \frac{\mu(A \cap V)}{\mu(C + V)} \ . \tag{6}$$

If $\Lambda \subset G$ is any (e.g. discrete) set and $\gamma := \gamma_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$ is the counting measure of Λ, then we get

$$\overline{D}^{\#}(\Lambda) := \overline{D}(\gamma_\Lambda; \mu) := \inf_{C \Subset G} \sup_{V \in \mathcal{B}_0} \frac{\#(\Lambda \cap V)}{\mu(C + V)} \ . \tag{7}$$

Theorem 5. *Let K be any convex body in $\mathbb{R}^d$ and normalize the Haar measure of $\mathbb{R}^d$ to be equal to the volume $|\cdot|$. Let ν be any measure with sigma algebra of measurable sets $\mathcal{S}$. Then we have*

$$\overline{D}(\nu; |\cdot|) = \overline{D}_K(\nu) \ . \tag{8}$$

The same statement applies also to $\mathbb{Z}^d$.

Remark 6. *In particular, we find that the asymptotic uniform upper density $\overline{D}_K(\nu)$ does not depend on the choice of K. For a direct proof of this one has to cover the boundary of a large homothetic copy of K by standard (unit) cubes, say, and after a tedious ϵ-calculus a limiting process yields the result. However, Theorem 1 elegantly overcomes these technical difficulties.*

Furthermore, we also introduce a second notion of density as follows.

Defintion 7. *Let G be a LCA group and $\mu := \mu_G$ be its Haar measure. If ν is another measure on G with the sigma algebra of measurable sets being $\mathcal{S}$, then we define*

$$\overline{\Delta}(\nu) := \overline{\Delta}(\nu; \mu) := \inf_{F \subset G,\, \#F < \infty} \sup_{V \in \mathcal{S} \cap \mathcal{B}_0} \frac{\nu(V)}{\mu(F+V)} \ . \tag{9}$$

In particular, if $A \subset G$ is Borel measurable and $\nu = \mu_A$ is the trace of the Haar measure on the set A, then we get

$$\overline{\Delta}(A) := \overline{\Delta}(\mu_A) := \overline{\Delta}(\mu_A; \mu) := \inf_{F \subset G,\, \#F < \infty} \sup_{V \in \mathcal{B}_0} \frac{\mu(A \cap V)}{\mu(F+V)} \ . \tag{10}$$

If $\Lambda \subset G$ is any (e.g. discrete) set and $\gamma := \gamma_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$ is the counting measure of Λ, then we get

$$\overline{\Delta}^{\#}(\Lambda) := \overline{\Delta}(\gamma_\Lambda; \mu) := \inf_{F \subset G,\, \#F < \infty} \sup_{V \in \mathcal{B}_0} \frac{\#(\Lambda \cap V)}{\mu(F+V)} \ . \tag{11}$$

The two definitions are rather similar, except that the requirements for $\overline{\Delta}$ refer to finite sets only. Because all finite sets are necessarily compact in an LCA group, (5) of Definition 4 extends the same infimum over a wider family of sets than (9) of Definition 7; therefore we get an obvious inequality. Also equality is obvious when G is discrete, for then comopact sets are just the finite sets. Using also a nice observation of Totik [17], we can even state the following comparison result.

Proposition 8. *Let G be any LCA group, with normalized Haar measure μ. Let ν be any measure with sigma algebra of measurable sets $\mathcal{S}$. Then we have*

$$\overline{\Delta}(\nu; \mu) \geq \overline{D}(\nu; \mu) \ . \tag{12}$$

Moreover, this inequality is an identity for discrete G, when μ becomes the counting measure $\#$. On the other hand for all non-discrete G there exists some probability measure ν – and even some set $A \subset G$ with $A \in \mathcal{B}$ and $\nu := \mu|A)$ – such that $\overline{\Delta}(\nu, \mu) > \overline{D}(\nu, \mu)$. In other words, $\overline{\Delta}$ and $\overline{D}$ coincide (for all ν) iff G is discrete.

There are several applications of the above new notion of a.u.u.d. in lca groups. We could extend several earlier results about the so-called packing-type estimates in connection to the so-called "Turán extremal problem", see in particular [11]. Also we could extend the following results.

Let us denote the upper density of $A \subset \mathbb{N}$ as $\overline{d}(A) := \limsup_{n\to\infty} A(n)/n > 0$ with $A(n) := \#(A \cap [1,n])$. Erdős and Sárközy (seemingly unpublished, but quoted in [4] and in [14]) observed the following.

Proposition 9 (Erdős-Sárközi)**.** *If the upper density $\overline{d}(A)$ of a sequence $A \subset \mathbb{N}$ is positive, then writing the positive elements of the sequence $D(A) := D_1(A) := A - A$ as $D(A) \cap \mathbb{N} = \{(0 <)d_1 < d_2 < \dots\}$ we have $d_{n+1} - d_n = O(1)$.*

This is analogous, but not contained in the following result of Hegyvári, obtained for σ-finite groups. An abelian group is called σ-finite (with respect to H_n), if there exists an increasing sequence of *finite* subgroups H_n so that $G = \cup_{n=1}^{\infty} H_n$. For such a group Hegyvári defines asymptotic upper density (with respect to H_n) of a subset $A \subset G$ as

$$\overline{d}_{H_n}(A) := \limsup_{n\to\infty} \frac{\#(A \cap H_n)}{\# H_n} \ . \tag{13}$$

Note that for finite groups this is just $\#(A \cap G)/\#G$. Hegyvári proves the following [4, Proposition 1].

Proposition 10 (Hegyvári)**.** *Let G be a σ-finite abelian group with respect to the increasing, exhausting sequence H_n of finite subgroups and let $A \subset G$ have positive upper density with respect to H_n. Then there exists a finite subset $B \subset G$ so that $A - A + B = G$. Moreover, we have $\#B \le 1/\overline{d}_{H_n}(A)$.*

Fürstenberg calls a subset $S \subset G$ in a topological Abelian (semi)group a *syndetic* set, if there exists a compact set $K \subset G$ such that for each element $g \in G$ there exists a $k \in K$ with $gk \in S$; in other words, in topological groups $\cup_{k\in K} Sk^{-1} = G$. Then he presents as Proposition 3.19 (a) of [2] the following.

Proposition 11 (Fürstenberg)**.** *Let $S \subset \mathbb{Z}$ with positive (upper) Banach density. Then $S - S$ is a syndetic set.*

In fact, our interest in the problem of the definition of a.u.u.d. in general LCA groups came from another problem, the so-called "Turán extremal problem" for positive definite functions. In that question some results, already known for classical situations like $\mathbb{R}^d$, $\mathbb{Z}^d$ or compact groups, could also be extended. For these questions we refer to [11].

Theorem 12. *If G is a LCA group with Haar measure μ, and $A \subset G$ has $\overline{\Delta}(A) = \overline{\Delta}(A;\mu) > 0$, then there exists a finite subset $B \subset G$ so that $A - A + B = G$. Moreover, we can find B with $\#B \le [1/\overline{\Delta}(A)]$.*

Remark 13. *We need a translation-invariant (Haar) measure, but not the topology or compactness.*

Corollary 14. *Let $A \subset \mathbb{R}^d$ be a (measurable) set with $\overline{\Delta}(A) > 0$. Then there exist $b_1, \dots, b_k$ with $k \le K := [1/\overline{\Delta}(A)]$ so that $\cup_{j=1}^k (A - A + b_j) = \mathbb{R}^d$.*

This is interesting as it shows that the difference set of a set of positive Banach density $\overline{\Delta}$ is necessarily rather large: just a few translated copies cover the whole space.

Corollary 15. *Let G be a LCA group and $S \subset G$ a set with positive a.u.u. density, i.e. $\overline{D}(S) > 0$, where here $\overline{D}(S) = \overline{D}(\mu|_S;\mu)$. Then the difference set $S - S$ is a syndetic set: moreover, the set of translations K, for which we have $G = S + K$, can be chosen not only compact, but even to be a finite set with $\#K \le [1/\overline{D}(S)]$ elements.*

This corollary is immediate, because $\overline{\Delta}(S) \ge \overline{D}(S)$ according to Proposition 8.

This indeed generalizes the proposition of Fürstenberg. Also this result contains the result of Hegyvári: for on σ-finite groups the natural topology is the discrete topology, whence the natural Haar measure is the counting measure, and so on σ-finite groups Corollary 15 and Theorem 12 coincides. Finally, this also generalizes and sharpens the Proposition of Erdős and Sárközy. Indeed, on $\mathbb{Z}$ or $\mathbb{N}$ we naturally have $\overline{\Delta}(A) = \overline{D}(A) \ge \overline{d}(A)$, so if the latter is positive, then so is $\overline{D}(A)$; and then the difference set is syndetic, with finitely many translates belonging to a translation set $K \subset \mathbb{N}$, say, covering the whole $\mathbb{Z}$. Hence $d_{n+1} - d_n - 1$ cannot exceed the maximal element of the finite set K of translations.

Theorem 16. *Let G be a LCA group and $S \subset G$ a set with a positive, (but finite) asymptotic uniform upper density, regarding now the counting measure of elements of S in the definition of Banach density, i.e. $\overline{D}(S) = \overline{D}(\#|_S;\mu) > 0$. Then the difference set $S - S$ is a syndetic set.*

Lemma 17 **(subadditivity).** *Let $\nu_0 = \sum_{j=1}^n \nu_j$ be a sum of measures, all on the common set algebra $\mathcal{S}$ of measurable sets. Then we have $\overline{D}(\nu_0, \mu) \le \sum_{j=1}^n \overline{D}(\nu_j, \mu)$.*

In particular, this holds for one given measure ν and a disjoint union of sets $A_0 = \cup_{j=1}^n A_j$, with $\nu_j := \nu|_{A_j}$, for $j = 0, 1, \dots, k$. If $\nu = \mu$, this gives $\overline{D}(\cup_{j=1}^n A_j) \le \sum_{j=1}^n \overline{D}(A_j)$.

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