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B-MEASURABILITY OF MULTIFUNCTIONS OF TWO VARIABLES

Abstract

We will consider Borel measurability of multifunctions defined on the product of two metrizable spaces with values in a perfectly normal topological space. We show that a multifunction of upper (lower) class α in the first variable and continuous in the second variable is of the lower (upper) class $\alpha + 1$. It turn out that replacing continuity in the second variable by semicontinuity our multifunction may be of no neither lower nor upper class α. We show some reinforcement of the semicontinuity with additional assumption which ensures the lower class $\alpha + 2$ for the multifunction.

Various results were published about Borel classification of multifunctions of one variable. Obviously each multifunction of two variables may be treated as a multifunction of a single variable. But in this case we have the possibility of formulation of hypotheses concerning the multifunction in terms of its sectionwise properties.

Let us begin with notations and basic definitions.

Let T and Z be two nonempty sets and let $\Phi : T \rightsquigarrow Z$ be a multifunction, i.e. $\Phi(t)$ is a nonempty subset of Z for $t \in T$.

Given $G \subset Z$, two counterimages of G may be defined:

$$\Phi^-(G) = \{ t \in T : F(t) \cap G \neq \emptyset\} \text{ and } \Phi^+(G) = \{ t \in T : F(t) \subset G\} \subset \Phi^-(G).$$

Note that $\Phi^+(G) \subset \Phi^-(G)$, so we will also say the big and the small counterimages of G, respectively.

The following relations hold between these counterimages:

$$\Phi^-(G) = T \setminus \Phi^+(Z \setminus G) \quad \text{and} \quad \Phi^+(G) = T \setminus \Phi^-(Z \setminus G).$$

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If $\phi : T \to Z$ is a function and a multifunction $\Phi : T \rightsquigarrow Z$ is given by

$$\Phi(t) = \{\phi(t)\},$$

then

$$\Phi^-(G) = \Phi^+(G) = \phi^{-1}(G).$$

Now let $(T, \mathcal{T}(T))$ and $(Z, \mathcal{T}(Z))$ be the topological spaces. $\mathcal{B}(T)$ will denote the σ-field of Borel subsets of T.

A multifunction $\Phi : T \rightsquigarrow Z$ is called *upper* (*lower*) *semicontinuous* at a point $t \in T$ if,

$\forall G \in \mathcal{T}(Z)(\Phi(t) \subset G \Rightarrow t \in \mathrm{Int}\Phi^+(G))$
$(\forall G \in \mathcal{T}(Z)(\Phi(t) \cap G \neq \emptyset \Rightarrow t \in \mathrm{Int}\Phi^-(G)))$.

A multifunction Φ upper (lower) semicontinuous at each point $t \in T$ is said to be *upper* (*lower*) *semicontinuous*; Φ is called *continuous* if it is both lower and upper semicontinuous.

It is easy to see that Φ is lower (upper) semicontinuous if and only if $\Phi^-(G) \in \mathcal{T}(T)$ $(\Phi^+(G) \in \mathcal{T}(T))$, whenever $G \in \mathcal{T}(Z)$.

Following Neubrunn [8], a multifunction $\Phi : T \rightsquigarrow Z$ is said to be *upper* (*lower*) *quasi-continuous* at a point $t \in T$ if, for each $G \in \mathcal{T}(Z)$ such that $t \in \Phi^+(G)$ $(t \in \Phi^-(G))$ and for any open neighborhood U of t, there exists a nonempty open set $V \subset U$ such that $V \subset \Phi^+(G)$ $(V \subset \Phi^-(G))$; Φ is said to be *upper* (*lower*) *quasi-continuous* if it is upper (lower) quasi-continuous at each $t \in T$.

Given any countable ordinal number $\alpha < \Omega$, we denote by:

$\Sigma_\alpha(T)$ – the additive class α of the Borel subsets of T,
$\Pi_\alpha(T)$ – the multiplicative class α of the Borel subsets of T.

Following Kuratowski [6], a multifunction $\Phi : T \rightsquigarrow Z$ is called *B-measurable of lower class* α (in brief *of lower class* α) if $F^-(G) \in \Sigma_\alpha(T)$, whenever $G \in \mathcal{T}(Z)$.

A multifunction $\Phi : T \rightsquigarrow Z$ is called *B-measurable of upper class* α (in brief *of upper class* α) if $F^+(G) \in \Sigma_\alpha(T)$, whenever $G \in \mathcal{T}(Z)$.

Note that Φ is of lower (upper) class α if and only if $\Phi^+(D) \in \Sigma_\alpha(T)$ $(\Phi^-(D) \in \Sigma_\alpha(T))$ for each closed set $D \subset Z$.

We will denote by

LB_α – the family of all multifunctions of the lower class α,
UB_α – the family of all multifunctions of the upper class α.

It is obvious, that $\Phi \in LB_0$ $(\Phi \in UB_0)$ if and only if Φ is lower (upper) semicontinuous.

The classes LB_α and UB_α for multifunctions of one variable have been studied by Kuratowski [7], Brisac [1], Hansell [3] and Garg [2], where many of the known results on real functions and also on lower and upper semicontinuous multifunctions have been extended to the general classes LB_α and UB_α.

Now we turn our attention to the well known result on real functions of two real variables. Lebesgue has shown that

Named Theorem. *Lebesgue Theorem. If a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of the Borel class α in the first and continuous in the second variable, then it is of the Borel class $\alpha + 1$.*

This theorem was extended by Kuratowski into the case of functions in metric spaces. We extend this result into the multivalued case in possible general spaces.

For a multifunction

$$F : X \times Y \rightsquigarrow Z$$

and $(x, y) \in X \times Y$ we define x-section and y-section of F as follows:

$F_x : Y \rightsquigarrow Z$ - x-section of F, given by $F_x(y) = F(x, y)$,
$F^y : X \rightsquigarrow Z$ - y-section of F, given by $F^y(x) = F(x, y)$.

Theorem 1. *Let (Y, d) be a separable metric space and let $(X, \mathcal{T}(X)), (Y, \mathcal{T}(Y))$ be perfectly normal topological spaces. If $F : X \times Y \rightsquigarrow Z$ is a compact valued multifunction such that all its y-sections are of the lower class α and all its x-sections are continuous, then F is of the upper class $\alpha + 1$.*

If we put in the above theorem upper semicontinuouity instead of the continuity, this result fails.

Example 2. Let $A \notin \mathcal{B}(\mathbb{R})$ and $B = \mathbb{R} \setminus A$.

Let $F : \mathbb{R} \times \mathbb{R} \rightsquigarrow \mathbb{R}$ be a multifunction defined by

$$F(x, y) = \begin{cases} [0, 1], & \text{if } x \neq y, \\ [0, 2], & \text{if } x = y \in A, \\ [-2, 1], & \text{if } x = y \in B. \end{cases}$$

Then all x-sections and y-sections of F are constant except of one point and the sections F_x and F^y are upper semicontinuous for each $x, y \in \mathbb{R}$. So

$\forall\, y \in \mathbb{R}\ F^y \in LB_1$ (since $F^y \in UB_0$) and

$\forall\, x \in \mathbb{R}\, F_x$ is upper semicontinuous.

But F is not of the upper class 2, because

$$F^+((-1, 3)) = \{(x, y) : F(x, y) \subset (-1, 3)\} = \mathbb{R}^2 \setminus \{(x, x) : x \in B\} \notin \mathcal{B}(\mathbb{R}^2).$$

Precisely, F is neither of any lower nor upper class α.

The next theorem is dual to Theorem 1.

Theorem 3. *Let (Y, d) be a separable metric space and let $(X, \mathcal{T}(X)), (Y, \mathcal{T}(Y))$ be perfectly normal topological spaces. If $F : X \times Y \rightsquigarrow Z$ is a multifunction such that all its y-sections are of the upper class α and all its x-sections are continuous, then F is of the lower class $\alpha + 1$.*

Just the same as in Theorem 1 if we put in the above theorem lower semicontinuouity instead of the continuouity, then this result fails, as the following example shows.

Example 4. Let A and B be as in Example 1 and let $F : \mathbb{R} \times \mathbb{R} \rightsquigarrow \mathbb{R}$ be a multifunction defined by

$$F(x, y) = \begin{cases} [-2, 2], & \text{if } x \neq y, \\ [-1, 0], & \text{if } x = y \in A, \\ [0, 1], & \text{if } x = y \in B. \end{cases}$$

Then all x-sections and y-sections of F are constant except of one point and the sections F_x and F^y are lower semicontinuous for each $x, y \in \mathbb{R}$. Therefore

$\forall\, y \in \mathbb{R}\ F^y \in UB_1$ (since $F^y \in LB_0$) and

$\forall\, x \in \mathbb{R}\, F_x$ is lower semicontinuous.

But F is not of the lower class 2, because

$$F^-((-2, 0)) = \{(x, y) : F(x, y) \cap (-2, 0) \neq \emptyset\} = \mathbb{R}^2 \setminus \{(x, x) : x \in B\} \notin \mathcal{B}(\mathbb{R}^2).$$

It follows from the above example that if all x-sections and y-sections of a multifunction F are both lower semicontinuous, then its behavior my be very strange.

One can strengthen the lower semicontinuity assumption of x-sections of F to ensure the lower class $\alpha + 2$ of F. Namely

Theorem 5. *Let $(X, \mathcal{T}(X))$ be a perfectly normal topological space. Let (Y, d) be a separable metric spaces and (Z, ρ) a Polish space.*

Suppose that a multifunction $F : X \times Y \rightsquigarrow Z$ with closed values has all y-sections of the lower class α and all x-sections lower semicontinuous and upper quasi-continuous. Then F is of the lower class $\alpha + 2$.

Next theorem is a multivalued analog of the Kempisty theorem (cp. [5]).

Theorem 6. *Let $(X, \mathcal{T}(X))$ be a topological space, $(Y, \mathcal{T}(Y))$ a metrizable one and $(Z, \mathcal{T}(Z))$ a perfectly normal topological space. If $F : X \times Y \rightsquigarrow Z$ is a* *compact-valued multifunction with lower semicontinuous y-section and upper semicontinuous x-section for each $(x, y) \in X \times Y$, then F is of the upper class 1. If moreover $(Z, \mathcal{T}(Z))$ is separable, then F is also of the lower class 1.*

It follows from Example 1 that there exists a compact-valued multifunction $F : \mathbb{R} \times \mathbb{R} \rightsquigarrow \mathbb{R}$, whose all y-sections are of the lower class 1 and all x-sections are of the upper class 1, but F is neither of any upper nor lower class 2. So, we can see that Theorem 4 cannot be generalized into the higher classes.

The following theorem shows that the topological spaces X and Y in the Theorem 4 cannot be entirely arbitrary.

Theorem 7. *Assume CH. There are topological spaces $(X, \mathcal{T}(X)), (Y, \mathcal{T}(Y))$ and $(Z, \mathcal{T}(Z))$ and a multifunction $F : X \times Y \rightsquigarrow Z$ with compact values whose all y-sections are lower semicontinuous and all x-sections are upper semicontinuous, but F is neither of any upper nor lower class α.*

PROOF. Let $X = Y = \mathbb{R}$ be the real line with the Euclidean topology, let $\mathcal{I}$ be the ideal of sets of the first category in $\mathbb{R}$ and let $A \subset \mathbb{R}$ be given. Following Hashimoto [4], we define

$$A^* = \{p \in \mathbb{R} : \forall U(p)\ U(p) \cap A \notin \mathcal{I}\},$$

where $U(p)$ is a neighborhood of p.

Defining the $*$-closure of A, i.e. $\widetilde{A}$, by

$$\widetilde{A} = A \cup A^*,$$

leads to a new topology on $\mathbb{R}$, called the $*$-topology or Hashimoto topology.

The set $\mathbb{R}$ has now two topologies. It is known (see [4]), that

- $G \subset \mathbb{R}$ is $*$-open if and only if G is the difference of an open set and a set belonging to $\mathcal{I}$.

Let $M \subset \mathbb{R} \times \mathbb{R}$ be a set such that all y-sections $M^y = \{x \in X : (x, y) \in M\}$ are countable and all x-sections $M_x = \{y \in Y : (x, y) \in M\}$ have countable complements. The set M exists by CH.

Note that M has not the Baire property. Otherwise M would be of the first category in $\mathbb{R}^2$ (since M^y are countable) but it is impossible by the Kuratowski-Ulam theorem.

Now let $Z = \mathbb{R}$ be the real line with the Euclidean topology and let $F : X \times Y \rightsquigarrow Z$ a multifunction given by

$$F(x, y) = \begin{cases} [-1, 3], & \text{if } (x, y) \in M, \\ [0, 1], & \text{if } (x, y) \in \mathbb{R}^2 \setminus M. \end{cases}$$

Note that if $G \subset \mathbb{R}$ is an open set, then $F_x^+(G)$ and $F^{y-}(G)$ are $*$-open (by ($\bullet$)). Thus

$\forall\, y \in Y \quad F^y$ is $*$-lower semicontinuous,

$\forall\, x \in X \quad F_x$ is $*$-upper semicontinuous.

On the other hand

$$F^-((2,3)) = F^-([2,3]) = M.$$

So, M is neither of any additive nor multiplicative class α, which finishes the proof. □

Last of all it may be well to add that if the multifunction F is point valued, then Theorems 1, 2 and 3 give the Lebesgue theorem and Theorem 4 is the Baire theorem on functions with continuous sections.

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