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Richard Mabry, Department of Mathematics, Louisiana State University in Shreveport, Shreveport, LA 71115-2399, U.S.A.
email: `Richard.Mabry@LSUS.edu`

RECENT "IMPROVEMENTS" OF THE LEBESGUE MEASURE

Theorem 1. *There exists a Banach measure μ on $\mathbb{R}^2$ such that if $A \subseteq \mathbb{R}^2$ is a set such that all the linear shades of A are equal to t, then the* (planar) *μ-shade of A is also equal to t.*

The construction of the measure μ in Theorem 1 is due to James W. Roberts. The necessary definitions follow.

Definition (Banach measure)**.** Let X denote either $\mathbb{R}$ or $\mathbb{R}^2$. If μ is an extension of the Lebesgue measure λ on X that is isometry-invariant and defined for all subsets of X, then μ is called a *Banach measure on X*.

Definition (shade)**.** Let X denote either $\mathbb{R}$ or $\mathbb{R}^2$.

(a) For a Banach measure μ on X, if $A \subseteq X$ and $\mu(A \cap J) = t\,\mu(J)$ for all bounded open $J \subset X$, we say that A *has μ-shade equal to t* and we write $\text{sh}_\mu(A) = t$.

(b) If $\text{sh}_\mu(A) = t$ for all Banach measures μ on X, then we say that A *has shade equal to t* and we write $\text{sh}(A) = t$.

When $X = \mathbb{R}$ (resp., $\mathbb{R}^2$) we refer to *linear* (resp., *planar*) measures or shades defined for certain subsets of X. In a natural way, for particular sets $A \subseteq \mathbb{R}^2$, we also consider (linear) shades of sets of the form $A \cap L$ when L is a line in $\mathbb{R}^2$.

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