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A DENSITY THEOREM IN MEASURE THEORY

Abstract

We exhibit conditions ensuring that any finitely additive function, which is defined on a Boolean algebra and takes values into a Hausdorff topological commutative group, is the pointwise limit of strongly continuous and s-bounded.

1 Introduction.

The aim of the present note is to announce some results concerning pointwise approximability of finitely additive functions, defined on a Boolean algebra and taking values into a Hausdorff topological commutative group, by means of strongly continuous and exhaustive finitely additive functions. Their proofs and other results on this matter can be found in [2].

Let $\mathcal{A}$ be a Boolean algebra and G a Hausdorff topological commutative group, both non-trivial.

Consider the group $a(\mathcal{A}, G)$ of all G-valued finitely additive functions on $\mathcal{A}$, equipped with the product topology τ_p, and let $csa(\mathcal{A}, G)$ denote its subgroup consisting of those functions $\mu \in a(\mathcal{A}, G)$ such that

▷ μ is exhaustive (or s-bounded), *i.e.* $\lim_k \mu(a_k) = 0$ for each disjoint sequence $(a_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$;

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▷ μ is strongly continuous, *i.e.* for any 0-neighborhood U in G there exists some finite partition $\{d_1, \ldots, d_n\} \subseteq \mathcal{A}$ of the maximal element of $\mathcal{A}$ such that $\mu(d_i \wedge a) \in U$ for all i and $a \in \mathcal{A}$.

We seek minimal conditions on the algebra $\mathcal{A}$ and the group G under which the denseness result

$$\overline{csa(\mathcal{A}, G)}^{\tau_p} = a(\mathcal{A}, G) \tag{1}$$

does hold true.

This problem was firstly investigated by K. P. S. Bhaskara Rao and M. Bhaskara Rao in the 1970's in the framework $G = \mathbb{R}$. In [1] they proved that whenever the Boolean algebra $\mathcal{A}$ is atomless then each finitely additive probability measure defined on $\mathcal{A}$ can be pointwise approximated by means of strongly continuous finitely additive probability measures, *i.e.* the set $csa(\mathcal{A}, \mathbb{R})$ is dense in $(sa(\mathcal{A}, \mathbb{R}), \tau_p)$.

Recently, Klimkim and Svistula [4] have furnished a similar result in the case that $G = (X, ||\cdot||_X)$ is a Banach space. That is, if $\mathcal{A}$ is atomless then each X-valued exhaustive finitely additive function on $\mathcal{A}$ is the pointwise limit of strongly continuous and exhaustive finitely additive functions; thus the set $csa(\mathcal{A}, X)$ is dense in $(sa(\mathcal{A}, X), \tau_p)$.

Our purpose is to treat the denseness problem (1) in fully generality, whatever Hausdorff topological commutative group is taken into account. In this general framework, the approach of [1, 4] does not work, since the group G is not required to be complete.

Our point of departure is a handy characterization of the denseness in $a(\mathcal{A}, G)$ of those sets which, loosely speaking, are closed under the operation of sum and suitable restrictions (Lemma 1). The set $csa(\mathcal{A}, G)$ fulfils such properties and the criteria can be used to derive a necessary condition on $\mathcal{A}$ for (1). Sufficient conditions both on $\mathcal{A}$ and on G for the validity of (1) are then described in Theorem

2 below.

2 Main Results.

Lemma 1. *Let $M \subseteq a(\mathcal{A}, G)$. Assume that the set M fulfils*

$$M + M \subseteq M, \qquad \mu_a \in M \qquad \textit{for all } \mu \in M,\ a \in \mathcal{A}, \tag{2}$$

where $\mu_a(b) := \mu(a \wedge b)$, $b \in \mathcal{A}$.
Then the set M is dense in $(a(\mathcal{A}, G), \tau_p)$ if, and only if, the set $M(a) := \{\mu(a) : \mu \in a(\mathcal{A}, G)\}$ is dense in G for all $a \in \mathcal{A} \setminus \{0\}$.

Lemma 1 can be used to reduce the problem of the denseness of a large class of sets in $a(\mathcal{A}, G)$ to corresponding denseness problems inside the group G. Note that, because of the continuity of the projection maps, the necessity condition always holds. Thus condition (2) is needed to ensure the converse.
Customary subsets of $a(\mathcal{A}, G)$ fulfils condition (2); in particular,

as a consequence of Lemma 1, one discovers that the set $ua(\mathcal{A}, G)$ consisting of all finite sums of ultrafilter measures is dense in $a(\mathcal{A}, G)$. Hence

$$\overline{ua(\mathcal{A}, G)}^{\tau_p} = \overline{fsa(\mathcal{A}, G)}^{\tau_p} = \overline{sa(\mathcal{A}, G)}^{\tau_p} = a(\mathcal{A}, G), \tag{3}$$

where $sa(\mathcal{A}, G) := \{\mu \in a(\mathcal{A}, G) : \mu$ is exhaustive$\}$ and $fsa(\mathcal{A}, G) := \{\mu \in sa(\mathcal{A}, G) : \mu(\mathcal{A})$ is finite$\}$.

Moreover, since the subgroup $csa(\mathcal{A}, G)$ fulfils (2) as well,

Lemma 1 entails that the validity of (1) forces the algebra $\mathcal{A}$ to be atomless.

In fact, any function belonging to $csa(\mathcal{A}, G)$ must be zero on atoms of $\mathcal{A}$, and both $\mathcal{A}$ and G are non-trivial.
In the light of this fact, the non-atomicity of $\mathcal{A}$ is our sole assumption on $\mathcal{A}$ approaching the question of whether the denseness result (1) does hold true. To formulate our assumption on the group G, and hence our answer to the denseness problem, recall

that a subgroup $H \subseteq G$ is said to be a *one-parameter subgroup of* G if there exists a continuous homomorphism $\phi : \mathbb{R} \to G$ such that $\phi(\mathbb{R}) = H$.

Theorem 2. *Let $\mathcal{A}$ be an atomless Boolean algebra. If the smallest closed subgroup of G which contains all one-parameter subgroups is G itself, then the set $csa(\mathcal{A}, G)$ is dense in $(a(\mathcal{A}, G), \tau_p)$, i.e. (1) holds true.*

Let us emphasize some of its consequences. Firstly consider the case of locally compact groups; for them [3, Theorem 25.20] tells us that the requirement on G of Theorem 2 is equivalent to that of connectedness of G. Henceforth

Corollary 3. *Let $\mathcal{A}$ be atomless. If G is locally compact and connected, then (1) holds true.*

Moreover, coupling the above-mentioned necessary condition on $\mathcal{A}$ for (1) and Theorem 2 provides the following strengthening of [4, Theorem 1].

Corollary 4. *Let G be a Hausdorff topological real vector space. Then (1) holds true if, and only if, $\mathcal{A}$ is atomless.*

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