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SOME CONSEQUENCES OF A DENSITY THEOREM IN MEASURE THEORY

Abstract

Let $\mathcal{A}$ be a Boolean algebra and G a complete Hausdorff topological commutative group. We present some properties which $\mathcal{A}$ and G have necessarily to fulfil for every finitely additive function from $\mathcal{A}$ to G to be the pointwise limit of strongly continuous and exhaustive finitely additive functions.

1 Introduction.

We discuss necessary conditions for pointwise approximability of finitely additive functions, defined on a Boolean algebra and taking values into a complete Hausdorff topological commutative group, by means of strongly continuous and exhaustive finitely additive functions. Combining these results with those presented in [2], we are able to furnish a precise characterization of this approximation problem. Proofs and other results on this matter can be found in [1]. For background notions we refer the reader to [2].

Let $\mathcal{A}$ be a Boolean algebra and G a complete Hausdorff topological commutative group, both non-trivial.
We assume that the denseness result

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$$\overline{csa(\mathcal{A}, G)}^{\tau_p} = a(\mathcal{A}, G) \tag{1}$$

does hold true. Here

$a(\mathcal{A}, G)$ is the group of all G-valued finitely additive functions on $\mathcal{A}$, equipped with the product topology τ_p, and $csa(\mathcal{A}, G)$ its subgroup consisting of those functions which are exhaustive and strongly continuous. In other words, our point of departure is that each G-valued finitely additive functions defined on $\mathcal{A}$ is the pointwise limit of strongly continuous and exhaustive finitely additive functions.

2 Main Results.

We firstly show that

I): The validity of (1) yields that the Boolean algebra $\mathcal{A}$ must be atomless.

In fact, any function belonging to $csa(\mathcal{A}, G)$ must be zero on atoms of $\mathcal{A}$, and both $\mathcal{A}$ and G are non-trivial.

Moreover

II): The validity of (1) implies that the set

$$c_o(G) := \{y \in G : \ y \text{ can be joined to } 0 \text{ by a path}\}$$

is dense in G. In particular, the group G must be connected.

On account of Theorem 2 and Corollary 3 in [2] we are able to deduce by *I)-II)* that

III): Whenever G is a locally compact group, then the following are equivalent:

- *i) (1) holds true;*
- *ii) $\mathcal{A}$ is atomless and the set $c_o(G)$ is dense in G;*
- *iii) $\mathcal{A}$ is atomless and G is connected.*

Let us emphasize that *III)* tells us that

for a locally compact group either (1) holds for *all* atomless Boolean algebras (if G is connected) or for *none* of them (if G is not connected).

We exhibit that such a dichotomy pertains to *any complete* group G, namely

IV): If (1) holds for some atomless algebra $\mathcal{A}$, then it holds for all of them.

In contrast to this, for $G = \mathbb{Q}$ the validity of (1) depends on $\mathcal{A}$ as the following examples show.

Example 1.- For any σ-algebra $\mathcal{A}$ the set $csa(\mathcal{A}, \mathbb{Q})$ is not dense in $a(\mathcal{A}, \mathbb{Q})$, *i.e.* (1) fails.

Example 2.- Let $\mathcal{A}$ be the algebra generated by the collection

$$\mathcal{F} := \{[(i-1)/2^n, i/2^n[: i, n \in \mathbb{N}, i \leq 2^n\}.$$

Then $csa(\mathcal{A}, \mathbb{Q})$ is dense in $a(\mathcal{A}, \mathbb{Q})$, *i.e.* (1) holds true.

We highlight that the characterizations of (1) described in *III)* however fail for merely complete groups, even tempting to replace the requirement of connectedness of G in *III)-iii)* by that of its arcwise connectedness, as shown in [1, Appendix 5].
What needed is, in fact, a stronger connected property of G, namely the denseness in G of its subset $c_1(G)$ consisting of all elements joined to 0 by some path $\gamma : [0,1] \to G$ satisfying the additional property that the series $\sum_{n=1}^{\infty}(\gamma(\beta_n) - \gamma(\alpha_n))$, where $([\alpha_n, \beta_n[)_{n \in \mathbb{N}}$ is any disjoint sequence in $[0,1[$, converges. In the special case $G = \mathbb{R}$, this means that the path γ must be of bounded variation.

In the case of merely complete groups our investigation yields that

V): Whenever G is complete, then the following are equivalent:

- *i) (1) holds true;*
- *ii) $\mathcal{A}$ is atomless and the set $c_1(G)$ -just defined- is dense in G.*

References

[1] P. Cavaliere, P. de Lucia and H. Weber, *Approximation of finitely additive functions valued into topological groups*, preprint.

[2] P. Cavaliere, P. de Lucia and H. Weber, *A density theorem in measure theory*, Proceedings of Summer Symposium in Real Analysis XXXV, Budapest, June 5-11, 2011.