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MEAGER-ADDITIVE SETS THROUGH THE PRISM OF FRACTAL DIMENSION

A subset of the line is called *strongly null* if its algebraic sum with any meager set does not cover the line. Merging the Galvin–Mycielski-Solovay Theorem and a result of Besicovitch, $X \subseteq \mathbb{R}$ is strongly null if and only if the Haudsorff dimension of every uniformly continuous image of X is zero.

A similar notion has been investigated since 1990's: A subset of the line (or any other Polish topological group) is *meager-additive* if its algebraic sum with any meager set is still meager. The only reasonable results obtained in the Cantor set, i.e. the topological group 2^ω. E.g. Shelah [2] provided a combinatorial characterization of meager-additive sets in 2^ω.

The problem of finding a characterization of meager-additive sets in terms of fractal dimension that would follow the pattern of the above mentioned characterization of strongly null sets has not been considered until the Windy City Symposium, where I reported such a description. With its aid I also partially solved a problem of Nowik and Weiss [1] on $\mathcal{E}$-additive sets.

Two major directions emerged: First, completing the missing part of the solution of Nowik-Weiss problem and second, a similar characterization of meager-additive sets in a more general context of, say, a compact Polish group.

In the sequel we let $\mathbb{G}$ be a locally compact metric Polish group. The group operation is denoted $+$, though $\mathbb{G}$ is not *a priori* supposed to be abelian. The algebraic sum of two sets $A, B \subseteq \mathbb{G}$ is defined by $A+B = \{a+b : a \in A, b \in B\}$.

Denote by $\mathcal{M}$ the ideal of meager sets in $\mathbb{G}$ and $\mathcal{E}$ the so called *intersection ideal*, i.e. the σ-ideal generated by closed Haar-null sets.

Definition 1. A set $X \subseteq \mathbb{G}$ is *$\mathcal{M}$-additive* if $\forall M \in \mathcal{M}\ X + M \in \mathcal{M}$, and *$\mathcal{E}$-additive* if $\forall E \in \mathcal{E}\ X + M \in \mathcal{E}$.

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Question 2 ([1]). Are $\mathcal{M}$-additive an $\mathcal{E}$-additive sets in 2^ω related?

Definition 3. Let X be a metric space. We consider the following modification of the usual Hausdorff dimension (denoted by $\dim_{\mathsf{H}}$). Let $X^\star$ denote the completion of X and define

$$\overline{\dim}_{\mathsf{H}} X = \inf\{\dim_{\mathsf{H}} K : K \text{ is } \sigma\text{-compact, } X \subseteq K \subseteq X^\star\}.$$

Part of the following theorem appeared in [4], namely (i)⇔(iii) and (i)⇒(ii). The implication (ii)⇒(i) is new, answering thus completely the question of Nowik and Weiss.

Theorem 4. *Let $X \subseteq 2^\omega$. The following are equivalent.*

(i) *X is $\mathcal{M}$-additive,*

(ii) *X is $\mathcal{E}$ additive*

(iii) *if $f : 2^\omega \to Y$ is a continuous mapping into a metric space Y, then $\overline{\dim}_{\mathsf{H}} f(X) = 0$.*

There are some consequences, e.g. a continuous image of an $\mathcal{M}$-additive set is $\mathcal{M}$-additive.

In 2009 Tomasz Weiss [3] proved that the canonical mapping T of 2^ω onto $[0,1]$ defined by $T(\langle x_n\rangle) = \sum_n 2^{-n-1}x_n$ preserves $\mathcal{M}$-additive sets in both directions. That made it possible to characterize $\mathcal{M}$-additive subsets of the line by (iii) of the above theorem. His proof, however, is very much bound to particular properties of the line and thus cannot be generalized beyond Euclidean spaces. A year later I succeded to find a combinatorial method that allowed to prove the following general theorem.

Theorem 5. *Let $\mathbb{G}$ be a locally compact Polish group admitting an invariant metric and $X \subseteq \mathbb{G}$. The following are equivalent.*

(i) *X is $\mathcal{M}$-additive,*

(ii) *if $f : \mathbb{G} \to Y$ is a continuous mapping into a metric space Y, then $\overline{\dim}_{\mathsf{H}} f(X) = 0$.*

(iii) *if $f : \mathbb{G} \to \mathbb{H}$ is a continuous mapping into a locally compact Polish group, then $\overline{\dim}_{\mathsf{H}} f(X) = 0$.*

Here are some consequences:

Corollary 6. *Let $\mathbb{G}$ and $\mathbb{H}$ be locally compact Polish groups admitting an invariant metric.*

(i) *If $X \subseteq \mathbb{G}$ and $Y \subseteq \mathbb{H}$ are $\mathcal{M}$-additive, then so is $X \times Y \subseteq \mathbb{G} \times \mathbb{H}$.*

(ii) *If $X \subseteq \mathbb{G}$ is $\mathcal{M}$-additive and $Y \subseteq \mathbb{H}$ is strongly null, then $X \times Y \subseteq \mathbb{G} \times \mathbb{H}$ is strongly null.*

(iii) *If $f : \mathbb{G} \to \mathbb{H}$ is continuous and $X \subseteq \mathbb{G}$ is $\mathcal{M}$-additive, then so is $f(X) \subseteq \mathbb{H}$.*

As to $\mathcal{E}$-additive sets, we have the following:

Theorem 7. *Let $\mathbb{G}$ be a locally compact Polish group and $X \subseteq \mathbb{G}$. If X is $\mathcal{M}$-additive, then it is $\mathcal{E}$-additive.*

There are two problems left that I consider interesting:

Question 8. Let $\mathbb{G}$ be a locally compact Polish group admitting an invariant metric. Is every $\mathcal{E}$-additive set in $\mathbb{G}$ $\mathcal{M}$-additive?

Question 9. Is the existence of invariant metric superfluous in theorem 5 and corollary 6?

References

[1] Andrzej Nowik and Tomasz Weiss, *On the Ramseyan properties of some special subsets of 2^ω and their algebraic sums*, J. Symbolic Logic **67** (2002), no. 2, 547–556. MR 1905154 (2003c:03087)

[2] Saharon Shelah, *Every null-additive set is meager-additive*, Israel J. Math. **89** (1995), no. 1-3, 357–376. MR 1324470 (96j:04002)

[3] Tomasz Weiss, *On meager additive and null additive sets in the Cantor space 2^ω and in $\mathbb{R}$*, Bull. Pol. Acad. Sci. Math. **57** (2009), no. 2, 91–99.

[4] Ondřej Zindulka, *Fractal measures vs. small sets of reals*, Real Anal. Exchange 2008, 32nd Summer Symposium Conference.