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VARIATIONAL MEASURES AND RADON-NIKODÝM PROPERTY

1 Introduction

Variational measures associated to a real valued function, or, in a more general setting, to a vector valued function are considered. Roughly speaking, given a function Φ defined on an interval of the real line it is possible to construct, using suitable families of intervals, a measure V_Φ which carries information about Φ. If Φ is a real valued function, then the σ-finiteness of the measure V_Φ implies the a.e. differentiability of Φ, while the absolute continuity of the measure V_Φ characterizes the functions Φ which are Henstock-Kurzweil primitives. The situation becomes more complicate if the functions take values in an infinite dimensional Banach space. If the Banach space has the Radon-Nikodým property, then it is possible to obtain properties similar to those of the real case. And it is surprising that by means of the variational measures it is possible to characterize the Banach spaces having the Radon-Nikodým property.

2 The Main Result

We denote by $\mathcal{I}$ the family of all nontrivial closed subintervals of $[0,1]$. A *partition in* $[0,1]$ is a finite collection of pairs $P = \{(I_1,t_1),\dots,(I_p,t_p)\}$, where $I_1,\dots,I_p$ are non overlapping intervals of $\mathcal{I}$ and $t_i \in I_i$, $i = 1,\dots,p$. Given $E \subset [0,1]$ we say that a partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ is *anchored in* E if $t_i \in E$ for each i. A *gauge* on $[0,1]$ is a positive function on $[0,1]$.

Given a gauge δ, we say that a partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ is *δ-fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1,\dots,p$.

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From now on X is a Banach space with norm $||\cdot||$.

Definition 1. A function $f:[0,1]\to X$ is said to be *variationally Henstock integrable*, if there exists an additive function $\Phi:\mathcal{I}\to X$, satisfying the following condition: given $\varepsilon>0$ there exists a gauge δ on $[0,1]$ such that

$$\sum_{i=1}^{p}\|f(t_i)|I_i|-\Phi(I_i)\|<\varepsilon\,,$$

for each δ-fine partition $\{(I_i,t_i):\ i=1,\dots,p\}$ in $[0,1]$. We set $\Phi(I)=(vH)\int_I f d\lambda$ and call the function Φ the *variational H-primitive of f*. By $vH([0,1],X)$ we denote the set of all vH-integrable functions $f\colon[0,1]\to X$.

Definition 2. Given an additive interval function $\Phi\colon\mathcal{I}\to X$, a gauge δ and a set $E\subset[0,1]$ we define

$$\operatorname{Var}(\Phi,\delta,E)=\sup\left\{\begin{array}{ll}\sum_{i=1}^{p}\|\Phi(I_i)\|: & \{(I_i,t_i):\ i=1,...,p\}\ \delta\text{−fine}\\ & \text{partition anchored on } E\end{array}\right\}.$$

Then we set

$$\mathrm{V}_\Phi(E)=\inf\{\operatorname{Var}(\Phi,\delta,E):\delta \text{ gauge on } E\}.$$

We call V_Φ the *variational measure generated by Φ*. It is known that V_Φ is a metric outer measure on $[0,1]$ (see [6]).

Theorem 1. (see [2]) *The following conditions are equivalent:*

1. *The Banach space X has the Radon-Nikodým property;*
2. *If $\Phi:\mathcal{I}\to X$ is BVG_* on $[0,1]$, then Φ is differentiable a.e. in $[0,1]$;*
3. *If V_Φ is σ-finite, then Φ is differentiable a.e. in $[0,1]$;*
4. *If $\mathrm{V}_\Phi\ll\lambda$, then Φ is differentiable a.e. in $[0,1]$;*
5. *If $\mathrm{V}_\Phi\ll\lambda$, then there exists $f\in vH([0,1],X)$ such that*

$$\Phi(I)=(vH)\int_I f(t)\,dt\,,\qquad \text{for every}\quad I\in\mathcal{I}\,.$$

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