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WHEN IS A FAMILY OF GENERALIZED MEANS A SCALE?

Quasi-arithmetic (QA) mean is defined for any continuous strictly monotone function $f\colon U \to \mathbb{R}$, U – an open interval. When $\underline{a} = (a_1, \ldots, a_n)$ is a sequence of points in U and $\underline{w} = (w_1, \ldots, w_n)$ is a sequence of weights ($w_i > 0$, $w_1 + \cdots + w_n = 1$), then the mean $\mathfrak{M} = \mathfrak{M}_f(\underline{a}, \underline{w})$ is defined by the equality $f(\mathfrak{M}) = \sum_{i=1}^n w_i f(a_i)$, directly generalizing the way power means are being defined.

This family of means was shown, by Kolmogorov in 1930 [1], to be very vast and ubiquitous. In fact, he proved that if a mean satisfied a tiny list of very natural axioms, then it had to be a QA mean for certain function f.

From the one side it is widely known, and by now classical, that there exists a close relationship between the comparability of two QA means $\mathfrak{M}_f$ and $\mathfrak{M}_g$ and the convexity of either the function $f \circ g^{-1}$, or $g \circ f^{-1}$. This condition is no doubt elegant, but not so useful. And it concerns only pairs of QA means – can be used only pair-wise.
From the other – and this is central in our approach, and can be neatly adopted to continuous families of QA means – in 1948 Mikusiński [2], in the first postwar issue of Banach and Steinhaus' renowned *Studia Mathematica*, put forward a very powerful tool in the theory of QA means. Namely, assuming $f, g \in \mathcal{C}^2(U)$ and f', g' vanishing nowhere in U,
$\Big(\mathfrak{M}_f \ge \mathfrak{M}_g$, with equality only when the vector $\underline{a}$ is constant$\Big)$ iff $\Big(\frac{f''}{f'} > \frac{g''}{g'}$ on a dense subset of $U\Big)$.

This has prompted us to consider an interesting question

- for what families $\{k_t \colon t \in I\}$ of functions sending U to $\mathbb{R}$, the mapping $I \ni t \mapsto k_t^{-1}\big(\sum_{i=1}^n w_i k_t(a_i)\big)$ is a continuous bijection between an open

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interval I and $(\min \underline{a}, \max \underline{a})$, for any fixed non-constant sequence $\underline{a}$ and any weights $\underline{w}$.

That is to say, when the family of QA means generated by functions $\{k_t : t \in I\}$ is a scale.

The precise assumptions in our Theorem read (all indicated derivatives are with respect to $x \in U$)
(i) k_t' does not vanish anywhere in U for every $t \in I$,
(ii) $I \ni t \mapsto \frac{k_t''(x)}{k_t'(x)} \in \mathbb{R}$ is increasing, 1–1 for x from a dense subset of U, and onto the image $\mathbb{R}$ for every $x \in U$.
It turns out that (i) and (ii) do for $\{k_t : t \in I\}$ to generate a scale of QA means.

This Theorem has, among nearly immediate corollaries, that (a) the family of power means is indeed a scale, and (b) so is the family of *radical* means.

References

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