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# FUNCTIONS OF $L^r$ BOUNDED VARIATION

We define the class of functions of $L^r$-bounded variation and show that on a bounded interval, this class consists of precisely the class of functions of bounded variation ($BV$). In the process we strengthen a theorem of P. M. and Y. Sagher.

In [1] we find the following:

**Theorem 1.** *Suppose $F \in L^r[0,1]$ is an approximately continuous function on $[0,1]$ and that $F([0,1]) \supseteq [0,c]$ for some $c > 0$. Then for any gauge function, $\delta$, there exists a finite collection of non-overlapping tagged intervals $\{(x_n,[u_n,v_n])\}$ subordinate to $\delta$ so that*

$$\sum_{n=1}^{N}\left(\frac{1}{v_n-u_n}\int_{u_n}^{v_n}\left|F(x)-F(x_n)\right|^r dx\right)^{\frac{1}{r}} > \frac{c}{12}. \tag{1}$$

We will show that $F$ need not be approximately continuous for the theorem to hold.

**Theorem 2.** *Suppose $F \in L^r[a,b]$ and that $F([a,b]) \supseteq \{c,d\}$ with $c < d$. Then for any gauge function, $\delta$, there exists a finite collection of non-overlapping tagged intervals $\{(x_n,[u_n,v_n])\}$ subordinate to $\delta$ so that*

$$\sum_{n=1}^{N}\left(\frac{1}{v_n-u_n}\int_{u_n}^{v_n}\left|F(x)-F(x_n)\right|^r dx\right)^{\frac{1}{r}} > \frac{d-c}{12}. \tag{2}$$

Without loss of generality, we will assume $a = c = 0$ and $b = d = r = 1$. We will first need some terminology.

**Definition 3.** *Let $0 < p \leq 1$ and let $E$ be a measurable subset of $[a,b]$. Let $x \in (a,b)$. We will say that $x$ is a point of $p$-lower density of $E$ if*

$$\liminf_{h\to 0^+}\frac{\lambda\left(E\cap(x-h,x+h)\right)}{2h} = p. \tag{3}$$

**Definition 4.** *Let $x \in [a,b)$. We will say that $x$ is a point of $p$-lower right-hand density of $E$ if*

$$\liminf_{h\to 0^+} \frac{\lambda(E\cap(x,x+h))}{h} = p. \tag{4}$$

*For convenience we will stipulate that if $b \in E$, then $b$ is a point of 1-lower right-hand density of $E$.*

**Definition 5.** *Let $x \in (a,b]$. We will say that $x$ is a point of $p$-lower left-hand density of $f$ if*

$$\liminf_{h\to 0^+} \frac{\lambda(E\cap(x-h,x))}{h} = p. \tag{5}$$

*For convenience we will stipulate that if $a \in E$, then $a$ is a point of 1-lower left-hand density of $E$.*

We start with an extension of a theorem of L. Gordon, found in [2].

**Lemma 6.** *Let $L$ and $R$ be nonempty disjoint measurable sets such that $[a,b] = L\cup R$, every point of $L$ is a point of $p$-lower left-hand density of $L$ for some $p \geq 2/3$, and every point of $R$ is a point of $p$-lower right-hand density of $R$ for some $p \geq 2/3$. Then every point of $R$ is to the right of every point of $L$.*

This allows us to prove the following.

**Lemma 7.** *Let $L$ and $R$ be nonempty disjoint measurable sets such that $[a,b] = L\cup R$. Then either there exists $x \in L$ such that*

$$\limsup_{h\to 0^+} \frac{\lambda(R\cap(x-h,x+h))}{2h} > \frac{1}{6} \tag{6}$$

*or there exists $y \in R$ such that*

$$\limsup_{h\to 0^+} \frac{\lambda(L\cap(y-h,y+h))}{2h} > \frac{1}{6}. \tag{7}$$

We do not claim that the constant 1/6 is optimal. The proof of Theorem 2 requires these lemmas.

## References


[1] . Musial and Y. Sagher, *The $L^r$ Henstock-Kurzweil Integral*, Studia Mathematica, **160**, (2004) 53–81.

[2] . Gordon, *Perron's Integral for Derivatives in $L^r$*, Studia Mathematica, **28,** (1967), 295–316.