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A CHARACTERIZATION OF THE WEAK RADON-NIKODÝM PROPERTY BY FINITELY ADDITIVE INTERVAL FUNCTIONS

1 Preliminaries

Let $[0,1]$ be the unit interval of the real line $\mathbb{R}$ equipped with the usual topology and the Lebesgue measure λ. We denote by $\mathcal{I}$ the family of all nontrivial closed subintervals of $[0,1]$ and by $\mathcal{L}$ the family of all Lebesgue measurable subsets of $[0,1]$.

Throughout this paper X is a Banach space. If μ is an outer measure on $[0,1]$, then by $\mu \ll \lambda$ we mean that $\lambda(E)=0$ implies $\mu(E)=0$. A mapping $\nu\colon \mathcal{L} \to X$ is said to be an X-*valued measure* if ν is countably additive in the norm topology of X. ν is said to be λ-*continuous* if $|E|=0$ implies $\nu(E)=0$. The *variation* of an X-valued measure ν is denoted by $|\nu|$. A function $f\colon [0,1] \to X$ is said to be *scalarly measurable* if for each $x^* \in X^*$ the real function x^*f is measurable.

A Banach space X has the weak Radon-Nikodým property (see [4] or [5, Theorem 11.3]) if and only if for every measure $\nu\colon \mathcal{L} \to X$ of σ-finite variation, that is absolutely continuous with respect to the Lebesgue measure, there exists a Pettis integrable function $f\colon [0,1] \to X$ such that

$$\nu(E) = \int_E f(t)\,dt, \quad \text{for every set } E \in \mathcal{L}. \tag{1}$$

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More information on Pettis-integrable functions can be found in [5] and [6]. A *partition in* $[0,1]$ is a finite collection of pairs $\mathcal{P}=\{(I_1,t_1),\ldots,(I_p,t_p)\}$, where $I_1,\ldots,I_p$ are non-overlapping subintervals of $[0,1]$ and $t_i\in I_i$, $i=1,\ldots,p$. Given a subset E of $[0,1]$, we say that the partition $\mathcal{P}$ is *anchored on* E if $t_i\in E$ for each $i=1,...,p$. If $\cup_{i=1}^{p} I_i=[0,1]$ we say that $\mathcal{P}$ is a *partition of* $[0,1]$. A *gauge* on $E\subset[0,1]$ is a positive function on E. For a given gauge δ, we say that a partition $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ is δ*-fine* if $I_i\subset(t_i-\delta(x_i),t_i+\delta(x_i))$, $i=1,\ldots,p$.

Definition 1. A function $f\colon[0,1]\to\mathbb{R}$ is said to be *Henstock-Kurzweil integrable*, (or HK*-integrable*), on $[0,1]$, if there exists $w\in\mathbb{R}$ with the following property: for every $\epsilon>0$ there exists a gauge δ on $[0,1]$ such that

$$\left|\sum_{i=1}^{p} f(t_i)|I_i|-w\right|<\varepsilon\,,$$

for each δ-fine partition $\mathcal{P}=\{(I_1,t_1),\ldots,(I_p,t_p)\}$ of $[0,1]$.
We set $w:=(HK)\int_0^1 f\,d\lambda$.

It is known that if $f\colon[0,1]\to\mathbb{R}$ is HK-integrable on $[0,1]$ and $I\in\mathcal{I}$, then $f\chi_I$ is also HK-integrable on $[0,1]$. We say in such a case that f is HK-integrable on I. We call the additive interval function $F(I):=(HK)\int_I f\,d\lambda$ the HK*-primitive* of f.

Definition 2. A function $f\colon[0,1]\to X$ is said to be *scalarly Henstock-Kurzweil integrable* if for each $x^*\in X^*$ the function x^*f is Henstock-Kurzweil integrable. A scalarly Henstock-Kurzweil integrable function f is said to be *Henstock-Kurzweil-Pettis integrable* (or *HKP-integrable*) if for each $I\in\mathcal{I}$ there exists $w_I\in X$ such that

$$\langle x^*,w_I\rangle=(HK)\int_I x^*f\,d\lambda\,,\quad\text{for every}\quad x^*\in X^*.$$

We call w_I the *Henstock-Kurzweil-Pettis integral* of f over I and we write $w_I:=(HKP)\int_I f\,d\lambda$. If $I=[a,b]$, then we write $(HKP)\int_a^b f\,d\lambda$ instead of $(HKP)\int_{[a,b]} f\,d\lambda$. □

We denote by $HKP([0,1],X)$ the set of all X-valued Henstock-Kurzweil-Pettis integrable functions on $[0,1]$ (functions that are scalarly equivalent are identified). More information on HKP-integrable functions can be found in [3].

It is known that the *HK*-primitive (resp. *HKP*-primitive) F of a function f is continuous (resp. weakly continuous, i.e. x^*F is continuous for every $x^*\in X^*$).

Definition 3. Let $F : [0,1] \to X$ be a function and G be a non-empty subset of $[0,1]$. If there is a function $F_p' : G \to X$ such that for each $x^* \in X^*$

$$\lim_{h\to 0} \frac{x^*F(t+h) - x^*F(t)}{h} = x^*(F_p'(t)) \ ,$$

for almost all $t \in G$, then F is said to be *pseudo-differentiable on* G (the exceptional sets depend on x^*), with a *pseudo-derivative* F_p'.

2 Variational Measures

Throughout the letter $\varPhi$ will denotes an arbitrary additive interval function $\varPhi\colon \mathcal{I} \to X$ that is identified with the point function $\varPhi(t) = \varPhi([0,t])$, $t \in [0,1]$.

Definition 4. Given $\varPhi\colon \mathcal{I} \to X$, a gauge δ and a set $E \subset [0,1]$ we define

$$\mathrm{Var}(\varPhi, \delta, E) := \sup \left\{ \begin{array}{ll} \sum_{i=1}^{p} \|\varPhi(I_i)\| : & \{(I_i, t_i) : \ i = 1, ..., p\} \ \delta\text{−fine} \\ & \text{partition anchored on } E \end{array} \right\} .$$

Then we set

$$\mathrm{V}_{\varPhi}(E) := \inf\{\mathrm{Var}(\varPhi, \delta, E) : \delta \text{ gauge on } E\}.$$

We call $\mathrm{V}_{\varPhi}$ the *variational measure generated by* $\varPhi$. It is known that $V_{\varPhi}$ is a metric outer measure on $[0,1]$ (see [7]). In particular, $V_{\varPhi}$ is a measure over all Borel sets of $[0,1]$.

Definition 5. We say that the variational measure $V_{\varPhi}$ is σ*-finite* if there is a sequence of (pairwise disjoint) sets F_n covering $[0,1]$ and such that $V_{\varPhi}(F_n) < \infty$, for every $n \in \mathbb{N}$.

Thomson (see [7, Theorem 3.15]) proved that $V_{\varPhi}$ has the so called measurable cover property, that is if $A \subset [0,1]$, then there exists $B \in \mathcal{L}$ such that $B \supset A$ and $V_{\varPhi}(B) = V_{\varPhi}(A)$. It follows from this that the sets F_n in the previous definition can be taken from $\mathcal{L}$.

Proposition 6. [1] If $V_{\varPhi} \ll \lambda$, then $\varPhi$ is continuous on $[0,1]$ and $V_{\varPhi}$ is σ-finite.

We recall that a function $\varPhi : [0,1] \to X$ is said to be BV_* on a set $E \subseteq [0,1]$ if $\sup \sum_{i=1}^{n} \omega(\varPhi(J_i)) < +\infty$, where the supremum is taken over all finite collections $\{J_1, ..., J_n\}$ of non overlapping intervals from $\mathcal{I}$ with end-points in E, and the symbol $\omega(\varPhi(J))$ stands for $\sup\{\|\varPhi(u) - \varPhi(z)\| : \ u, \ z \in J\}$. The function $\varPhi$ is said to be BVG_* on $[0,1]$ if $[0,1] = \bigcup_n E_n$ and $\varPhi$ is BV_* on each E_n.

Proposition 7. [1] *V_Φ is σ-finite if and only if Φ is BVG_* on $[0,1]$.*

The following theorem is the main result:

Theorem 8. [1] *Let X be a Banach space. Then the following conditions are equivalent:*
(i) X has the weak Radon-Nikodým property;
(ii) If $\Phi : \mathcal{I} \to X$ is BV_* on $[0,1]$, then Φ is pseudo-differentiable on $[0,1]$;
(iii) If $\Phi : \mathcal{I} \to X$ is BVG_* on $[0,1]$, then Φ is pseudo-differentiable on $[0,1]$;
(iv) If V_Φ is σ-finite, then Φ is pseudo-differentiable on $[0,1]$;
(v) If $V_\Phi \ll \lambda$, then Φ is pseudo-differentiable on $[0,1]$;
(vi) If $V_\Phi \ll \lambda$, then Φ is pseudo-differentiable on $[0,1]$, $\Phi'_p \in HKP([0,1],X)$ and

$$\Phi(I) = (HKP)\int_I \Phi'_p \, d\lambda\,, \qquad \text{for every} \quad I \in \mathcal{I}\,;$$

(vii) If $V_\Phi \ll \lambda$, then there exists $f \in HKP([0,1],X)$ such that

$$\Phi(I) = (HKP)\int_I f \, d\lambda\,, \qquad \text{for every} \quad I \in \mathcal{I}.$$

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