**Real Analysis Exchange**
Summer Symposium 2011, pp. 85–91

V. A. Skvortsov,* Department of Mathematics, Moscow State University, Moscow 119991, Russia; and Institute of Mathematics, Casimirus the Great University, Bydgoszcz 85-072, Poland. email: vaskvor2000@yahoo.com

# QUASI-MEASURE IN HARMONIC ANALYSIS AND ITS INTEGRAL REPRESENTATION

**Abstract**

We consider a role which the notion of quasi-measure plays in the theory of Walsh and Haar series and in harmonic analysis on zero-dimensional groups or, more generally, on zero-dimensional metric spaces.

The notion of quasi-measure appeared first in the theory of Walsh and Haar series. Let $S_{2^{\mathbf{k}}}$ be the partial sums of Walsh or Haar series (here $2^{\mathbf{k}}$ stand for $(2^{k_1}, \ldots, 2^{k_m})$). The integral

$$\int_{I^{(\mathbf{k})}} S_{2^{\mathbf{k}}}$$

where $I^{(\mathbf{k})}$ is a dyadic interval of rank $k$ either in $[0,1]^m$ or in $G^m$, where $G$ is the dyadic Cantor group, defines an additive interval function $\psi(I)$ (*quasi-measure*) on the family $\mathcal{I}_d$ of all dyadic intervals. Since the sum $S_{2^{\mathbf{k}}}$ is constant on each $I^{(\mathbf{k})}$ (in the interior of $I^{(\mathbf{k})}$ in the case of $[0,1]^m$) we get

$$S_{2^{\mathbf{k}}}(\mathbf{x}) = \frac{1}{|I^{(\mathbf{k})}|} \int_{I^{(\mathbf{k})}} S_{2^{\mathbf{k}}} = \frac{\psi(J^{(\mathbf{k})})}{|I^{(\mathbf{k})}|} \tag{1}$$

for any interior point $\mathbf{x} \in I^{(\mathbf{k})}$. This formula reduces the problem of recovering the coefficients of a series to the one of recovering the quasi-measure from its derivative.

---

Mathematical Reviews subject classification: Primary: 26A39, 42C10; Secondary: 43A70
Key words: quasi-measure, zero-dimensional metric space, integral representation of functional, Henstock-Kurzweil integral, Haar and Walsh series
*The research for this paper was supported by grants RFFI 11-01-00321a and NSh-3252.2010.1.

**Proposition 1.** *Let some integration process $\mathcal{A}$ be given which produces an additive integral on $\mathcal{I}_d$. Let the $\mathcal{I}_d$-interval function $\psi$ be defined for a Walsh or Haar series by* (*1*) *Then this series is the Fourier series of an $\mathcal{A}$-integrable function $f$ if and only if $\psi(I) = (\mathcal{A})\int_I f$ for any dyadic interval $I$.*

Formula (1) gives a relation between convergence of the series and differentiability of the quasi-measure only at the points with dyadic irrational coordinates. At other points we have to introduce various types of continuity, which are implied by various types of convergence of the series. For example, rectangular convergence at a point implies the *local Saks continuity*: $\lim_{\mu(I)\to 0,\, x\in I} \psi(I) = 0$. $\rho$-regular convergence implies another type of continuity. Assuming differentiability of a quasi-measure and its continuity in a certain sense we can recover it from its derivative by Henstock-Kurzweil or Perron type integrals (see [2] and [4] for details). As an example of a recent result obtained by the method of quasi-measure, we mention the following uniqueness theorem.

**Theorem** (M.Plotnikov)**.** *1). For any $\rho \in (\sqrt{2}/2, 1]$ there exists a non-trivial double Haar series which is $\rho$-regular convergent to zero everywhere on the unit square.*

*2). If $\rho \in (0, \sqrt{2}/2)$ then $\emptyset$ is $U$-set for $\rho$-regular convergence.*

For $\rho$-regular convergent Walsh series (with $\rho$ close to 1) the problem of uniqueness is open even in the case of cubic convergence.

In a similar way a quasi-measure can be generated by a series with respect to characters of a *zero-dimensional* compact abelian group $G$. Topology in such a group is known (see [1]) to be given by a chain of subgroups

$$G = G_0 \supset G_1 \supset \ldots \supset G_n \supset \ldots,$$

with $\{0\} = \bigcap_{n=0}^{+\infty} G_n$. Denote by $K^n$, $n \geq 0$ any coset of $G_n$ and call it $\mathcal{B}$-*interval*. For a fixed $g \in G$, let $K^n(g)$ be the coset of $G_n$ such that $g \in K^n(g)$, i.e., $K^n(g) = g + G_n$. For each $g \in G$ we have $\{g\} = \bigcap_n K^n(g)$. Let $\mu$ be the Haar measure on $G$, normalized so that $\mu(G) = 1$.

Let $\Gamma$ be the *dual group* of $G$, i.e., the group of *characters* of the group $G$. $\Gamma$ is a discrete abelian group with respect to the pointwise multiplication of characters. It can be represented as a sum of an increasing sequence of subgroups of finite order:

$$\Gamma_0 \subset \Gamma_{-1} \subset \Gamma_{-2} \subset \ldots \subset \Gamma_{-n} \subset \ldots$$

Then $\Gamma = \cup_{i=0}^{+\infty} \Gamma_{-i}$ and $\cap_{i=0}^{+\infty} \Gamma_{-i} = \{\gamma^{(0)}\}$ where $\gamma^{(0)}(g) = 1$ for all $g \in G$. For each $n \geq 0$ the group $\Gamma_{-n}$ is the annulator of $G_n$, i.e., $\Gamma_{-n} = G_n^{\perp} := \{\gamma \in \Gamma : \gamma(g) = 1 \text{ for all } g \in G_n\}$.

If $\gamma \in \Gamma_{-n}$ then $\gamma$ is constant on each coset $K^n$ of $G_n$.

We define a convergence of the series $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ at a point $g$ as the convergence of its partial sums of the form

$$S_n(g) := \sum_{\gamma\in\Gamma_{-n}} a_\gamma \gamma(g)$$

when $n \to \infty$. Now the *quasi-measure* associated with the series $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ is defined on each coset $K^n$ by

$$F(K^n) := \int_{K^n} S_n(g) d\mu. \tag{2}$$

$F$ is an additive function on the family of all $\mathcal{B}$-intervals. Since the sum $S_n$ is constant on each $K^n$, the definition of $F(K^n)$ implies

$$S_n(g) = \frac{F(K^n(g))}{\mu(K^n(g))}.$$

Once again the problem of recovering the coefficients of the series $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ from its sum is equivalent to the one of recovering the quasi-measure from its derivative. Indeed we have

**Theorem 2.** *A series $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ is the $\mathcal{A}$-Fourier series of some $\mathcal{A}$-integrable function $f$ if the quasi-measure $F$ associated by (2) with this series coincides on each $\mathcal{B}$-interval $K$ with the indefinite integral $(\mathcal{A})\int_K f d\mu$.*

The problem can be solved by a suitable Henstock-type integral which is defined below in a more general setting. It turns out that the group structure is not essential in this theory and we can formulate our problem using a zero-dimensional metric space instead of a group.

Let a sequence $\{C_n\}_{n=1}^{\infty}$ of covers of a zero-dimensional compact metric space $X$ be given such that

(a) for each fixed $n$, elements $K_j^{(n)}$ of $C_n$, are disjoint and clopen;

(b) each element of $C_n$ is properly contained in some element of $C_{n-1}$, for $n \geq 2$;

(c) $C_1 = \{X\}$.

Let $K(n,x)$ be the (unique) element $K_{j(n,x)}^{(n)}$ of $C_n$ such that $x \in K_{j(n,x)}^{(n)}$. A sequence $\{K(n,x)\}_n$ is defined for each $x$ so that $\bigcap_n K(n,x) = \{x\}$.

We assume that a Borel probability measure $\mu$ is given on $X$. So $\sum_{j=1}^{m(n)} \mu(K_j^{(n)}) = 1$. We denote $\mathcal{I} = \bigcup_{n=1}^{\infty} C_n$ and refer to elements of $\mathcal{I}$ as $\mathcal{B}$-intervals.

We introduce the set of *$\mathcal{B}$-polynomials*

$$P(X) = span\{\chi_K : K \in \mathcal{I}\}.$$

$P(X)$ is a dense linear subspace of $C(X)$. We call elements of $P(X)^\star$, the linear space dual of $P(X)$, *quasi-measures*. It is easy to understand that in the case of a zero-dimensional group this new notion of quasi-measure can be interpreted in the previous way.

If $S$ is a quasi-measure and $g \in P(X)$ we denote by $(S, g)$ the value of $S$ at $g$.

In particular a function $f$ of $L^1(\mu)$ defines a quasi-measures if we put its value at $g \in P(X)$ to be

$$(f, g) = (f d\mu, g) := \int_X f g \, d\mu.$$

A quasi-measure can be defined by specifying $(S, \chi_K)$ for each $K \in \bigcup_{n=0}^{\infty} C_n$.

If $S$ is a quasi-measure, $x$ an element of $X$, and $n \geq 0$, we define the $n$-th partial sum of the "Fourier series" of $S$ at $x$ to be

$$s_n(S, x) = s_n(S)(x) := (S, \chi_{K(n,x)})/\mu(K(n, x)).$$

For a fixed sequence $\{C_n\}_{n=1}^{\infty}$ and the measure $\mu$, we define a *derivation basis* $\mathcal{B}$ in $X$ as the family of all *basis sets*

$$\beta_\nu := \{(I, x) : x \in X, I = K(n, x), n \geq \nu(x)\}.$$

where $\nu$ runs over the set of all function $\nu : X \to \mathbb{N}$

A *$\beta_\nu$-partition* is a finite collection $\pi$ of elements of $\beta_\nu$, where the distinct elements $(I', x')$ and $(I'', x'')$ in $\pi$ have $I'$ and $I''$ disjoint. If $L \in \mathcal{I}$ and $\bigcup_{(I,x)\in\pi} I = L$ then $\pi$ is called $\beta_\nu$-partition *of* $L$.

A Henstock type integral in this setting can be defined as follows (see [3]).

**Definition 1.** Let $L \in \mathcal{I}$. A real-valued function $f$ on $L$ is said to be *Henstock integrable with respect to basis* $\mathcal{B}$ (or *$H_{\mathcal{B}}$-integrable)* on $L$, with *$H_{\mathcal{B}}$-integral* $A$, if for every $\varepsilon > 0$, there exists a function $\nu : L \mapsto \mathbb{N}$ such that for any $\beta_\nu$-partition $\pi$ of $L$ we have:

$$\left| \sum_{(I,x)\in\pi} f(x)\mu(I) - A \right| < \varepsilon.$$

We denote the integral value $A$ by $(H_{\mathcal{B}}) \int_L f$.

If a function $f$ is $H_{\mathcal{B}}$-integrable on $L$ and $f = h$ almost everywhere, then $h$ is also $H_{\mathcal{B}}$-integrable and their integrals coincide. This justifies the following generalization of the previous definition.

**Definition 2.** A real-valued function $f$ defined almost everywhere on $L \in \mathcal{I}$ is said to be $H_{\mathcal{B}}$-*integrable* on $L$, with integral value $A$, if the function

$$f_1(x) := \begin{cases} f(x), & \text{where } f \text{ is defined,} \\ 0, & \text{otherwise} \end{cases}$$

is $H_{\mathcal{B}}$-integrable on $L$ to $A$ in the sense of Definition 1.

If $f$ is $H_{\mathcal{B}}$-integrable on $L \in \mathcal{I}$ then it is $H_{\mathcal{B}}$-integrable also on any $\mathcal{B}$-interval $K \subset L$. So we can define the indefinite $H_{\mathcal{B}}$-integral $F(K) = (H_{\mathcal{B}})\int_K f d\mu$. It is an additive $\mathcal{B}$-interval function on the set of all $\mathcal{B}$-intervals $K \subset L$.

**Definition 3.** Given a real-valued set function $F$ on $\mathcal{I}$ we define the *upper* and the *lower* $\mathcal{B}$-*derivative* at a point $x$, with respect to the basis $\mathcal{B}$ and measure $\mu$, as

$$\overline{D}_{\mathcal{B}}F(x) := \limsup_{n\to\infty} \frac{F(K(n,x))}{\mu(K(n,x))}$$

and

$$\underline{D}_{\mathcal{B}}F(x) := \liminf_{n\to\infty} \frac{F(K(n,x))}{\mu(K(n,x))},$$

respectively. If $\overline{D}_{\mathcal{B}}F(x) = \underline{D}_{\mathcal{B}}F(x)$, then $F$ is $\mathcal{B}$-*differentiable* at the point $x$ with $\mathcal{B}$-*derivative* $D_{\mathcal{B}}F(x)$ being this common value.

**Theorem 3.** *If a function $f$ is $H_{\mathcal{B}}$-integrable on a $\mathcal{B}$-interval $L$ then the indefinite $H_{\mathcal{B}}$-integral $F(I) = (H_{\mathcal{B}})\int_I f$ as an additive function on the set of all $\mathcal{B}$-subintervals of $L$, is $\mathcal{B}$-differentiable almost everywhere on $L$ and*

$$D_{\mathcal{B}}F(x) = f(x) \quad \text{a.e. on} \quad L.$$

The above theorem holds for every measure $\mu$. Some other properties, in particular continuity of the indefinite integral, depend on the fact whether $\mu$ is non-atomic or not.

**Definition 4.** A real-valued set function $F$ defined on $\mathcal{I}$ is $\mathcal{B}$-*continuous* at a point $x$ if $\lim_{n\to\infty} F(K(n,x)) = 0$.

Any non-atomic measure $\mu$ on $X$ is $\mathcal{B}$-continuous at any point. The indefinite $H_{\mathcal{B}}$-integral on $L \in \mathcal{I}$ is $\mathcal{B}$-continuous at each point of $L$ if the measure $\mu$ is non-atomic.

The following results is related to an integral representation of a quasi-measure by $H_{\mathcal{B}}$-integral.

**Theorem 4.** *Suppose that $S$ is a quasi-measure and the measure $\mu$ is non-atomic. Let $s_n(S)$ and an $H_{\mathcal{B}}$-integrable function $f$ satisfy the inequality*

$$\liminf_{n\to\infty} s_n(S,x) \le f(x) \le \limsup_{n\to\infty} s_n(S,x)$$

*everywhere on $X$ except on a countable set, where*

$$\liminf_{n\to\infty}(S,\chi_{K(n,x)}) \le 0 \le \limsup_{n\to\infty}(S,\chi_{K(n,x)})$$

*holds. Then the sequence $s_n(S,x)$ is convergent to $f$ almost everywhere and the quasi-measure $S$ can be represented as*

$$(S,g) = (f,g) = (H_{\mathcal{B}})\int_X fg$$

*for each $\mathcal{B}$-polynomial $g \in P(X)$.*

**Corollary 5.** *Suppose that $S$ is a quasi-measure and the measure $\mu$ is non-atomic. Let $s_n(S)$ and an $H_{\mathcal{B}}$-integrable function $f$ satisfy the inequality*

$$\liminf_{n\to\infty} s_n(S,x) \le f(x) \le \limsup_{n\to\infty} s_n(S,x)$$

*almost everywhere and the condition*

$$\limsup |s_n(S,x)| < \infty$$

*everywhere on $X$ except on a countable set, where*

$$\liminf_{n\to\infty}(S,\chi_{K(n,x)}) \le 0 \le \limsup_{n\to\infty}(S,\chi_{K(n,x)})$$

*holds. Then the sequence $s_n(S,x)$ is convergent to $f$ almost everywhere and the quasi-measure $S$ can be represented as*

$$(S,g) = (f,g) = (H_{\mathcal{B}})\int_X fg$$

*for each $g \in P(X)$.*

**Theorem 6.** *Let $S$ be a quasi-measure and let $s_n(S)$ converge to a function $f$ everywhere on $X$ outside of a set $E \cup M$ such that $\mu(E) = 0$ with*

$$\limsup_{n\to\infty} |s_n(S,x)| < \infty$$

*everywhere on* $E$ *and* $M$ *is a countable set where*

$$\lim_{n\to\infty}(S, \chi_{K(n,x)}) = 0.$$

*Then* $f$ *is* $H_{\mathcal{B}}$*-integrable on* $X$ *in the sense of Definition 2 and* $S$ *can be identified with* $f$ *so that for each* $g \in P(X)$

$$(S, g) = (f, g) = (H_{\mathcal{B}}) \int_X fg.$$

Some of the above results are to be published in [5].

## References

[1] G.N. Agaev, N. Ya. Vilenkin, G.M. Dzhafarli, A.I. Rubistein, *Multiplicative system of functions and harmonic analysis on zero-dimensional groups,* Baku (1981)(In Russian).

[2] M. Plotnikov, *Quasi-measures, Hausdorff p-measures and Walsh and Haar series*, Izvestiya RAN **74**, no. 4, (2010), 157-188.

[3] V.A.Skvortsov, F. Tulone, *Representation of quasi-measure by Henstock-Kurzweil type integral on a compact-zero dimensional metric space*, Georgian Mathematical Journal **16**, no. 3, (2009), 575-582.

[4] V. Skvortsov, F. Tulone *Two-dimensional dyadic Henstock- and Perron-type integrals in the theory of Haar and Walsh series*, preprint.

[5] V. Skvortsov, F. Tulone, *Henstock Type Integral on a Compact Zero-Dimensional Metric Space and Representation of Quasi-Measure*, Vestnik of Moscow Univ. (to be published).