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HENSTOCK- AND PERRON-TYPE INTEGRAL ON A COMPACT ZERO-DIMENSIONAL METRIC SPACE

1 Introduction

A derivation basis and a Henstock-Kurzweil type integral ($H_{\mathcal{B}}$-integral) with respect to this basis on a compact zero-dimensional metric space X were introduced in [6] and [7].

We discuss here also two Perron-types integrals ($P_{\mathcal{B}}$-integral and $P^0_{\mathcal{B}}$-integral) with respect to this basis.

The aim of this paper is to prove the following chain of inclusions

$$L \subset P^0_{\mathcal{B}} \subset P_{\mathcal{B}} = H_{\mathcal{B}} \subset P^0_{\mathcal{B}}.$$

These results are known for the corresponding integrals with respect to the usual interval basis on the real line. See [2] and [3].

2 Preliminaries

Let a sequence $\{C_n\}_{n=1}^{\infty}$ of covers of a compact zero-dimensional metric space X be given such that

(a) for each fixed n, elements $K_j^{(n)}$ of C_n, are disjoint and clopen;

(b) each element of C_n is properly contained in some element of C_{n-1}, for $n \geq 2$;

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(c) $C_1 = \{X\}$;

(d) $\bigcup_{n=1}^{\infty} C_n$ is a base for the topology of X.

Let $C_n = \left\{K_j^{(n)}\right\}_{j=1}^{m(n)}$. For each $x \in X$ and $n \in \mathbb{N}$ let $K(n,x)$ be the (unique) element $K_{j(n,x)}^{(n)}$ of C_n such that $x \in K_{j(n,x)}^{(n)}$. A sequence $\{K(n,x)\}_n$ is defined for each x so that (in view of (d))

$$\bigcap_n K(n,x) = \{x\}.$$

We assume that a Borel probability measure μ is given on X. So, for each fixed n we have

$$\sum_{j=1}^{m(n)} \mu(K_j^{(n)}) = 1.$$

This measure can be extended in a usual way to be a complete measure on X. It is known that this type of measure being a completion of a Borel measure is regular (see [1]).

3 Definitions

For a fixed sequence $\{C_n\}_{n=1}^{\infty}$ and the measure μ, we define a *derivation basis* $\mathcal{B}$ in X as the family of all sets

$$\beta_\nu := \{(I,x) : x \in X, I = K(n,x), n \geq \nu(x)\}$$

where ν runs over the set of all functions $\nu : X \to \mathbb{N}$

Let $\mathcal{I} = \bigcup_{n=1}^{\infty} C_n$ and refer to elements of $\mathcal{I}$ as *$\mathcal{B}$-intervals.*

This basis has all the usual properties of a general derivation basis (see [3], [8]).

Definition 1. A *β_ν-partition* is a finite collection π of elements of β_ν, where distinct elements (I',x') and (I'',x'') in π have I' and I'' disjoint. If $L \in \mathcal{I}$ and $\bigcup_{(I,x)\in\pi} I \subset L$ then π is called β_ν-partition *in* L, if $\bigcup_{(I,x)\in\pi} I = L$ then π is called β_ν-partition *of* L.

Our a basis $\mathcal{B}$ has the *partitioning property* , i.e.,

i) for each finite collection $I_0, I_1, \ldots, I_n$ of $\mathcal{B}$- intervals with $I_1, \ldots, I_n \subset I_0$ and $I_i,\ i = 1, 2, \ldots, n$, being pairwise disjoint, the difference $I_0 \setminus \bigcup_{i=1}^n I_i$ can be expressed as a finite union of pairwise disjoint $\mathcal{B}$-intervals;

ii) for each $\mathcal{B}$-interval L and for any $\beta_\nu \in \mathcal{B}$ there exists a β_ν-partition of L.

For a set $E \subset X$ and $\beta_\nu \in \mathcal{B}$ we write

$$\beta_\nu(E) := \{(I, x) \in \beta_\nu : I \subset E\}$$

and

$$\beta_\nu[E] := \{(I, x) \in \beta_\nu : x \in E\}.$$

Definition of a Henstock-Kurzweil type integral:

Definition 2. Let $L \in \mathcal{I}$. A real-valued function f on L is said to be *Henstock-Kurzweil integrable with respect to the basis* $\mathcal{B}$ (or *$H_\mathcal{B}$-integrable*) on L, with *$H_\mathcal{B}$-integral* A, if for every $\varepsilon > 0$, there exists a function $\nu : L \to \mathbb{N}$ such that for any β_ν-partition π of L we have:

$$\left| \sum_{(I,x)\in\pi} f(x)\mu(I) - A \right| < \varepsilon.$$

We denote the integral value A by $(H_\mathcal{B}) \int_L f$.

If a function f is $H_\mathcal{B}$-integrable on X and $f = h$ almost everywhere, then h is also $H_\mathcal{B}$-integrable and their integrals coincide. So:

Definition 3. A real-valued function f defined almost everywhere on $L \in \mathcal{I}$ is said to be *$H_\mathcal{B}$-integrable* on L, with integral value A, if the function

$$f_1(g) := \begin{cases} f(g), & \text{where } f \text{ is defined,} \\ 0, & \text{otherwise,} \end{cases}$$

is $H_\mathcal{B}$-integrable on L to A in the sense of Definition 2.

If f is $H_\mathcal{B}$-integrable on $L \in \mathcal{I}$ then it is $H_\mathcal{B}$-integrable also on any $\mathcal{B}$-interval $K \subset L$. So we can define the indefinite integral $F(K) = (H_\mathcal{B})\int_K f d\mu$. The indefinite $H_\mathcal{B}$-integral F is an additive $\mathcal{B}$-interval function on the set of all $\mathcal{B}$-intervals $K \subset L$.

Definition 4. Given a real-valued set function F on $\mathcal{I}$ we define the *upper* and *lower $\mathcal{B}$-derivative* at a point x, with respect to the basis $\mathcal{B}$ and measure μ, as

$$\overline{D}_\mathcal{B} F(x) := \limsup_{n\to\infty} \frac{F(K(n,x))}{\mu(K(n,x))}$$

and

$$\underline{D}_{\mathcal{B}}F(x) := \liminf_{n\to\infty} \frac{F(K(n,x))}{\mu(K(n,x))},$$

respectively. If $\overline{D}_{\mathcal{B}}F(x) = \underline{D}_{\mathcal{B}}F(x)$, then F is $\mathcal{B}$-differentiable at the point x with $\mathcal{B}$-*derivative*, $D_{\mathcal{B}}F(x)$ being this common value.

Theorem 1. *If a function f is $H_{\mathcal{B}}$-integrable on a $\mathcal{B}$-interval L then the indefinite $H_{\mathcal{B}}$-integral $F(I) = (H_{\mathcal{B}}) \int_I f$ as an additive function on the set of all $\mathcal{B}$-subintervals of L, is $\mathcal{B}$-differentiable almost everywhere on L and*

$$D_{\mathcal{B}}F(x) = f(x) \quad \text{a.e.} \quad \text{on} \quad L.$$

The above theorem holds for every measure μ. Some other properties, in particular continuity of the indefinite integral, depend on the fact whether μ is non-atomic or not.

Definition 5. A real-valued set function F defined on $\mathcal{I}$ is $\mathcal{B}$-*continuous* at a point x if

$$\lim_{n\to\infty} F(K(n,x)) = 0.$$

Any non-atomic measure μ on X is $\mathcal{B}$-continuous at any point.

The indefinite $H_{\mathcal{B}}$-integral on $L \in \mathcal{I}$ is $\mathcal{B}$-continuous at each point of L if the measure μ is non-atomic.

Now we define a Perron type integral with respect to the basis $\mathcal{B}$.

Definition 6. Let f be a point function on X. A $\mathcal{B}$-interval function M (resp. m) is called a $\mathcal{B}$-*major* (resp. $\mathcal{B}$-*minor*) function of f on X if it is superadditive (resp. subadditive) and the lower (resp. upper) $\mathcal{B}$-derivative satisfies the inequality

$$\underline{D}_{\mathcal{B}}M(x) \geq f(x) \;\; (\text{resp.} \;\; \overline{D}_{\mathcal{B}}m(x) \leq f(x))$$

for all $x \in X$. A function f is said to be $P_{\mathcal{B}}$-*integrable*, if it has at least one $\mathcal{B}$-*major* and one $\mathcal{B}$-*minor* function and

$$-\infty < \inf_M \{M(X)\} = \sup_m \{m(X)\} < +\infty$$

where “inf” is taken over all $\mathcal{B}$-major function M and “sup” is taken over all $\mathcal{B}$-minor function m. The common value is denoted by $(P_{\mathcal{B}}) \int_X f$ and is called $P_{\mathcal{B}}$-integral of f on X.

For any $\mathcal{B}$-*major* function M and for any $\mathcal{B}$-*minor* function m we have $M(X) \geq m(X)$. This implies the correctness of the previous definition.

In the same way we can define $P_{\mathcal{B}}$-integral on any $\mathcal{B}$-interval.

If in the above definition we assume all the $\mathcal{B}$- major and $\mathcal{B}$-minor functions to be $\mathcal{B}$-continuous we obtain the definition of $P^0_{\mathcal{B}}$-*integral*. It is clear that $P^0_{\mathcal{B}}$-integral is included in $P_{\mathcal{B}}$-integral.

Let f be a $P_{\mathcal{B}}$-integrable function on X. Since f is also integrable on each $\mathcal{B}$-interval $I \subset X$, we can define the indefinite integrals $F(I) = (P_{\mathcal{B}}) \int_I f$ and $F(I) = (P^0_{\mathcal{B}}) \int_I f$. The indefinite integral F is an additive $\mathcal{B}$-interval function on $\mathcal{I}$ in both cases.

In the standard way we can check that $H_{\mathcal{B}} = P_{\mathcal{B}}$.

Similarly to the case of $H_{\mathcal{B}}$-integral the above Perron-type integrals can be defined in the case of functions defined only almost everywhere.

To compare $H_{\mathcal{B}}$-integral with $P^0_{\mathcal{B}}$-integral we shall use the notion of variation.

Let F be an additive set function on $\mathcal{I}$, E an arbitrary fixed subset of X, and A a $\mathcal{B}$- interval. For a fixed $\beta_\nu \in \mathcal{B}$, we set

$$V_\nu(A) = V(E, F, \beta_\nu, A) :=$$
$$\sup \left\{ \sum_{(I,g)\in\pi} |F(I)| : \pi \subset \beta_\nu[E] \cap \beta_\nu(A) \right\}$$

and we call it the β_ν-*variation of the function* F *on* $E \cap A$. In case $E \cap A = \emptyset$ we define $V_\nu(A) = 0$. For a fixed E, $V_\nu(A)$ is a non negative and superadditive interval function.

4 Main results.

The next theorem can be proved for our basis $\mathcal{B}$ in a similar way as an analogous result in [5] for the particular case of zero-dimensional group.

Theorem 2. *Let F be a $\mathcal{B}$-continuous additive function defined on $\mathcal{I}$ with a finite value $V_\nu(X)$. Then for a fixed $E \subset X$ and a fixed function $\nu : X \to \mathbb{N}$ the $\mathcal{B}$-interval function $V_\nu(A) = V(E, F, \beta_\nu, A)$ is $\mathcal{B}$-continuous at each point $x \in X$.*

The above theorem can be used to construct a $\mathcal{B}$-continuous major and minor functions for an $H_{\mathcal{B}}$-integrable function.

Theorem 3. *Suppose that the measure μ on X is non-atomic and a real-valued function f is $H_{\mathcal{B}}$-integrable on X, with F being its indefinite $H_{\mathcal{B}}$-integral.*

Then for any $\varepsilon > 0$ there exist a $\mathcal{B}$-continuous $\mathcal{B}$-major function M and a $\mathcal{B}$-continuous $\mathcal{B}$-minor function m of f such that

$$M(X) - F(X) < \varepsilon \ \textit{and} \ F(X) - m(X) < \varepsilon.$$

The idea of proof is the following:

Let $E = X \setminus C$ where $C = \{x \in X : D_{\mathcal{B}}F(x) = f(x)\}$, $\mu(E) = 0$. For this E and for $\varepsilon > 0$ we can choose ν such that

$$V_\nu(X) = V(E, F, \beta_\nu, X) < \varepsilon$$

By Theorem 2, the $\mathcal{B}$-interval function $V_\nu(A) = V(E, F, \beta_\nu, A)$ is $\mathcal{B}$-continuous at each point x of X. Then the functions

$$M(A) = F(A) + V_\nu(A) \ \text{and} \ m(A) = F(A) - V_\nu(A)$$

are the major and minor function we are looking for.

Using the previous theorem we obtain for the case of non-atomic measure μ the following scheme

$$P^0_{\mathcal{B}} \subset P_{\mathcal{B}} = H_{\mathcal{B}} \subset P^0_{\mathcal{B}}.$$

So we have

Theorem 4. *If measure μ is non-atomic, then $H_{\mathcal{B}}$-integral is equivalent to both $P_{\mathcal{B}}$- and $P^0_{\mathcal{B}}$-integral.*

In particular we have got that for non-atomic measure μ and for our basis $\mathcal{B}$, the $P^0_{\mathcal{B}}$- and $P_{\mathcal{B}}$-integral are equivalent. We note that for a general basis the problem about the equivalence of the above Perron type integrals is still open.

Now to complete the chain of inclusions we prove for non atomic measure $L \subset P^0_{\mathcal{B}}$.

We need the following version of Vitali-Caratheodory theorem which is proved in [4, Chapter III, Theorem 7.6] for functions defined on $\mathbb{R}^m$ but the same proof can be used for functions defined on any compact metric space with a regular measure on it.

Theorem 5. *Given a real-valued summable function f on a compact metric space X with a regular measure μ and any $\varepsilon > 0$, there exist a summable lower semi-continuous function l and a summable upper semi-continuous function u such that*

$$l(x) \geq f(x) \geq u(x) \ \textit{at each point} \ x \in X,$$

$$\int_X [l(x) - f(x)]d\mu < \varepsilon \ \textit{and} \ \int_X [f(x) - u(x)]d\mu < \varepsilon.$$

Theorem 6. *Let f be a real-valued summable function on a zero-dimensional compact metric space X with non-atomic regular measure μ. Then for any ε there exists a $\mathcal{B}$-continuous major function M and a $\mathcal{B}$-continuous minor function m such that $M(X) - \int_X f d\mu < \varepsilon$ and $\int_X f d\mu - m(X) < \varepsilon$.*

For the proof it is enough to put $M(I) = \int_I l d\mu$ and $m(I) = \int_I u d\mu$ where l and u are taken from the Vitali-Caratheodory theorem.

5 Quasi measure.

We introduce the set of $\mathcal{B}$-*polynomials*

$$P_X = span\{\chi_K : K \in \mathcal{I}\}.$$

$P(X)$ is a dense linear subspace of $C(X)$. We call elements of $P(X)^\star$, the linear space dual of $P(X)$, *quasi-measures*. If S is a quasi-measure and $g \in P(X)$ we denote by (S, g) the value of S at g.

A quasi-measure can be defined by specifying (S, χ_K) for all $K \in \bigcup_{n=0}^{\infty} C_n$.

If S is a quasi-measure, x an element of X, and $n \geq 0$, we define the nth partial sum of the "Fourier series" of S at x to be

$$s_n(S, x) = s_n(S)(x) := (S, \chi_{K(n,x)})/\mu(K(n, x)).$$

The result related to an integral representation of a quasi-measure by the L-integral:

Theorem 7. *Suppose that S is a quasi-measure. Let $s_n(S)$ and an L-integrable function f satisfy the inequality*

$$\liminf_{n\to\infty} s_n(S, x) \leq f(x) \leq \limsup_{n\to\infty} s_n(S, x)$$

everywhere on X. Then the sequence $s_n(S, x)$ is convergent to f a.e. and the quasi measure S can be represented as $(S, g) = (f, g) = (L)\int_X fg$ for each $\mathcal{B}$-polynomial $g \in P(X)$.

The main results of this paper will be published in [9].

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