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COANALYTIC TRANSFINITE CONSTRUCTIONS

Definition 1. *A two-point set is a subset of the plane, which intersects every line in exactly two points.*

Mazurkiewicz showed the existence of a two-point set with straightforward transfinite recursion. P. Erdős asked that whether a two-point set can be a Borel set.

However this question is still open, Miller in [3] proved that under certain set theoretic assumptions (namely $V = L$) one can construct a coanalytic two-point set. In the same article Miller proves the consistent existence of a coanalytic MAD family and Hamel basis. The author proves the statement solely for two-point sets and the proof uses deep set theoretical tools. In several later published articles there are references on Miller's method without clear proof.

Our aim was to make precise and prove a "black box" condition which could be easily applied without the set theoretical machinery.

At first we define the Turing reducibility. Let M be $\mathbb{R}^n$, 2^ω or ω^ω.

Definition 2. *Suppose that $x, y \in M$. We say that x is Turing reducible to y if there exists a Turing machine, which computes x with the oracle y. The notation is $x \leq_T y$. Let us say that $A \subset M$ is Turing cofinal, if for every $x \in \mathbb{R}$ there exists a $y \in A$ such that $x \leq_T y$.*

Now let us formulate our statement. Roughly speaking the theorem will state that if we have transfinite recursion and the selection algorithm (F) is nice enough, moreover if in every step the set of possible choices is large enough (in Turing sense), then we can produce a coanalytic set. In most cases we have to choose form each class an element, for example in the construction

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of two-point set it has to intersect every line, so we also use a set of parameters (typically every real).

If $S \subset X \times Y$ and $x \in X$ we denote the x-section of S (i. e. $\{y \in Y : (x,y) \in S\}$) with S_x. ω denotes the first infinite ordinal, ω_1 is the first uncountable ordinal. For a set M the set of countable sequences of elements of M is denoted by $M^{\leq\omega}$. Note that if M is a polish space then so is $M^{\leq\omega}$.

Theorem 1. *(V=L) Suppose that $F \subset M^{\leq\omega} \times \mathbb{R} \times M$ a coanalytic set and for all $p \in \mathbb{R}$, $A \in M^{\leq\omega}$ $F_{(A,p)}$ is Turing cofinal. Then there exists an enumeration of $\mathbb{R} = \{p_\alpha : \alpha < \omega_1\}$ and a coanalytic set $X = \{x_\alpha : \alpha < \omega_1\}$, such that for all $\alpha < \omega_1$ $x_\alpha \in F_{((x_n)_{n\in\omega}, p_\alpha)}$, where $(x_n)_{n\in\omega}$ is a certain enumeration of $\{x_\beta : \beta < \alpha\}$.*

So $F_{(A,p)}$ is the set of possible choices, if we have chosen the sequence A and the next parameter is p.

Remark. *Theorem 1 implies the consistent existence of coanalytic Hamel basis, MAD family, two-point set and an uncountable subset of the plane which intersects every C^1 curve in countably many points.*

References

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