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CHAOS AMONG SELF-MAPS OF AN n -DIMENSIONAL MANIFOLD AND OF THE CANTOR SPACE

1 Introduction

Chaos is a concept of considerable interest and importance in dynamical systems. We focus on three main notions of chaos: positive topological entropy, chaos according to Devaney and chaos according to Li-Yorke. We provide an overview of these types of chaos for self-maps of n -dimensional manifolds and of the Cantor space, and of the generic behavior, reporting well-known and recent results.

Here is how we proceed:

Section 1 is dedicated to preliminary definitions, with particular attention to minimal sets and adding machines, and to basic results. The study of the structure of ω -limit sets is fundamental in order to understand the chaotic nature of the generating function and how it is affected by slight perturbations. Adding machines (also referred to as solenoids or odometers) occur abundantly as ω -limit sets. They are all Cantor sets topologically. They are a type of infinite minimal sets. Minimal sets were defined by G.D. Birkhoff [5] and are very interesting and important in dynamical systems.

Agronsky, Bruckner and Laczkovich show in [3] that given a generic continuous self-map f of the unit interval $I = [0, 1]$ there is a residual set of points x in I for which the ω -limit set $\omega(x, f)$ is a Cantor set. Using a much different approach, Lehning extends these results to continuous self-maps of any compact n -dimensional manifold [23]. In [25] Steele goes a step further by showing

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that, on the interval, a generic ordered pair (x, f) gives rise to an ω -limit set generated by an adding machine. Continuing this line of enquiry, in [13], it is shown that there is a residual set of points (x, f) in $M \times \mathcal{C}(M, M)$, where M is an n -dimensional manifold or the Cantor space, for which $\omega(x, f)$ is a particular α -adic adding machine of type ∞ , and that if M is an n -dimensional manifold with the fixed point property a generic element of $\mathcal{C}(M, M)$ generates uncountably many distinct α -adic adding machines for every possible α . Hence, adding machines, that are very nice dynamical systems and far from being chaotic in any sense whatsoever, appear very frequently. Therefore, chaos cannot be detected pointwise.

In Section 2 we recall the three notions of chaos, positive topological entropy, Devaney chaos and Li-Yorke chaos, and their (eventual) relations.

We end with Section 3, where we report some recent results concerning Devaney chaos, obtained jointly with U.B. Darji.

2 Some preliminary notions and basic results

Let X be a compact metric space. By $C(X) = C(X, X)$ we denote the space of all continuous functions from X into X , endowed with the sup norm $\|\cdot\|$.

Definition 2.1. *Let $x \in X$ and $f \in C(X)$. The trajectory of x under f is*

$$\gamma(x, f) \equiv \{f^k(x)\}_{k \geq 0}$$

and the ω -limit set of f at x is

$$\omega(x, f) \equiv \bigcap_{m \geq 0} \overline{(\cup_{k \geq m} f^k(x))}.$$

Topological conjugation

([7]: page 18) Let X and Y be metric spaces, and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be continuous maps. The maps f and g are said to be *topologically conjugate* if there exists a homeomorphism $h : X \rightarrow Y$ of X onto Y such that $h \circ f = g \circ h$.

The concept of topological conjugation is fundamental since topologically conjugate maps have essentially the same properties.

Minimal sets

([7]: page 91) Let $f \in C(X)$. A subset of X is *minimal* for f if it is non-empty, closed and invariant and no proper subset has these three properties.

Theorem 2.2. ([7]: Lemma V.1) Let $f \in C(X)$. A non-empty set $T \subseteq X$ is minimal if and only if $\omega(x, f) = T$ for every $x \in T$.

Remark 2.3. A finite set is a minimal set if and only if it is a periodic orbit.

Adding machines

The terminology is borrowed from [8]. If $\alpha \in (N \setminus \{1\})^N$, set

$$\Delta_\alpha = \prod_{i=1}^{\infty} Z_{\alpha(i)},$$

where $Z_k = \{0, \dots, k-1\}$.

Instead of the usual coordinate-wise addition, we add two elements of Δ_α with “carry over” to the right. More precisely,

$$(x_1, x_2, \dots) \text{ and } (y_1, y_2, \dots) \text{ in } \Delta_\alpha \Rightarrow \\ (x_1, x_2, \dots) + (y_1, y_2, \dots) = (z_1, z_2, \dots),$$

where

$$z_1 = (x_1 + y_1) \bmod(\alpha(1))$$

and, in general,

$$z_i = x_i + y_i + \epsilon_{i-1} \bmod(\alpha(i))$$

where

$$\epsilon_{i-1} = \begin{cases} 0 & \text{if } x_{i-1} + y_{i-1} + \epsilon_{i-2} < \alpha(i-1) \\ 1 & \text{otherwise.} \end{cases}$$

If we let f_α be the “+1” map, that is

$$f_\alpha(x_1, x_2, \dots) = (x_1, x_2, \dots) + (1, 0, 0, \dots),$$

then $(\Delta_\alpha, f_\alpha)$ is a dynamical system known in various contexts as a α -adic solenoid, adding machine or odometer.

Definition 2.4. [8] Let $\alpha \in (N \setminus \{1\})^N$. Let M_α from the set of primes into $N \cup \{\infty\}$ be defined as follows. For each prime p , let

$$M_\alpha(p) = \sum_{i=1}^{\infty} n(i),$$

where $n(i)$ is the largest power of p which divides $\alpha(i)$.

In the following theorem Block and Keesling characterize adding machines up to topological conjugacy [8].

Theorem 2.5. [8] *Let $\alpha, \beta \in (N \setminus \{1\})^N$. Then f_α and f_β are topologically conjugate if and only if $M_\alpha = M_\beta$.*

Definition 2.6. ([13], [14]) *We call odometers of type ∞ those odometers associated with those α for which $M_\alpha(p) = \infty$ for all p .*

The next result, due to Block and Keesling [8], is essential to the proofs of Theorem 2.8 and Theorem 2.9 stating the “abundance” of adding machines in dynamical systems.

Theorem 2.7. [8] *Let $\alpha, \beta \in (N \setminus \{1\})^N$, $m_i = \alpha(1)\alpha(2)\dots\alpha(i)$, for each i , and $f : X \rightarrow X$ a continuous map of a compact topological space X . Then f is topologically conjugate to f_α if and only if (1), (2), and (3) hold.*

(1) *For each positive integer i , there is a cover P_i of X consisting of m_i pairwise disjoint, nonempty, clopen sets which are cyclically permuted by f .*

(2) *For each positive integer i , P_{i+1} partitions P_i .*

(3) *If $W_1 \supset W_2 \supset W_3 \supset \dots$ is a nested sequence with $W_i \in P_i$ for each i , then $\bigcap_{i=1}^\infty W_i$ consists of a single point.*

Moreover, in this case statement (4) also holds.

(4) *X is metrizable and if $\text{mesh}(P_i)$ denotes the maximum diameter of an element of the cover P_i , then $\text{mesh}(P_i) \rightarrow 0$ as $i \rightarrow \infty$.*

The abundance of adding machines

The two following results can be viewed as an extension of [3], [23] and [25], as the structure of the adding machines generated by a generic continuous self-map of M where M is a n -manifold or the Cantor space is analyzed.

Theorem 2.8. [13] *The set $\{(x, f) \in M \times C(M, M) : \omega(x, f)$ is an odometer of type $\infty\}$ is residual in $M \times C(M, M)$.*

Theorem 2.9. [13] *Let M be an n -manifold with the fixed point property. A generic $f \in C(M, M)$ has the property that for each $\alpha \in (N \setminus \{1\})^N$ there are continuum many pairwise disjoint ω -limit sets of f such that are topologically conjugate to $(\Delta_\alpha, f_\alpha)$.*

Remark 2.10. *Adding machines are very nice dynamical systems and far from being chaotic in any sense whatsoever. As Theorem 2.8 and Theorem*

2.9 show, a generic map exhibits adding machine-like behavior at most points, so we have that chaos is a phenomenon which cannot be captured pointwise.

3 Chaos

Three notions of chaos and their (eventual) relations

We now focus our attention on three important notions of chaos: positive topological entropy, chaos according to Devaney and chaos according to Li-Yorke.

Topological entropy

([9]; [16]; [12]) Let (X, d) be a compact metric space and $f : X \rightarrow X$ be a continuous map. For each natural number n , a new metric d_n is defined on X by the formula

$$d_n(x, y) = \max\{d(f^i(x), f^i(y)) : 0 \leq i < n\}.$$

A subset E of X is said to be (n, ϵ, f) -separated if each pair of distinct points of E is at least ϵ apart in the metric d_n . Denote by $N(n, \epsilon, f)$ the maximum cardinality of an (n, ϵ, f) -separated set. The *topological entropy of the map f* is defined by

$$\text{ent}(f) = \lim_{\epsilon \rightarrow 0} \left(\limsup_{n \rightarrow \infty} \frac{1}{n} \log N(n, \epsilon, f) \right).$$

We recall [10] that in the above we may use \liminf , i.e.,

$$\text{ent}(f) = \lim_{\epsilon \rightarrow 0} \left(\liminf_{n \rightarrow \infty} \frac{1}{n} \log N(n, \epsilon, f) \right).$$

Devaney Chaos

([15]; [18]; [12]) The system (X, f) , where X has no isolated points, is chaotic according to Devaney if the mapping f is transitive and periodic points of f are dense in X . Often this definition is somewhat rigid so one only requires these properties on a subsystem, i.e., there is a perfect set $Y \subseteq X$ such that $f|_Y$, the restriction of f to Y , is Devaney chaotic.

Li-Yorke chaos

([24]; [12]) The system (X, f) is chaotic according to Li-Yorke if there is an uncountable scrambled set S , i.e., for all distinct points $x, y \in S$ we have that

$$\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0$$

and

$$\limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > 0.$$

On the unit interval the following implications hold:

$$\begin{aligned} & \text{ent}(f) > 0 \\ \Leftrightarrow & f \text{ Devaney chaotic on a subsystem} \\ \Rightarrow & f \text{ Li-Yorke chaotic} \end{aligned}$$

In the case of a complete metric space with no isolated points, Huang and Ye [20] have proved

$$f \text{ Devaney chaotic} \Rightarrow f \text{ Li-Yorke chaotic}$$

and Blanchard, Glasner, Kolyada, and Maass [6] have settled a longstanding problem by proving the following implication

$$h(f) > 0 \Rightarrow f \text{ Li-Yorke chaotic}$$

Remark 3.1. *In the general setting of compact metric spaces there are no implications between positive topological entropy and Devaney chaos on a subsystem.*

Recently, many authors have considered both: the space $\mathcal{H}(X)$, the set of all homeomorphisms of X and the space $C(X)$. As in these spaces the Baire category theorem holds, they have asked what type of dynamical behavior is exhibited by a generic homeomorphism or a generic continuous map. Some of them are recalled below. (We also mention that generic homeomorphisms are deeply studied in the monograph by Akin, Hurley and Kennedy [2]).

Remark 3.2. *The situation on the interval is well-known. Of course, a homeomorphism of the interval is not chaotic in any sense. A generic continuous self-map of the interval has infinite topological entropy and therefore it is chaotic on a Devaney subsystem and chaotic according to Li-Yorke [7].*

Theorem 3.3. [26] *A generic homeomorphism of a manifold of dimension at least 2 has the property that it has infinite topological entropy. The same result holds for a generic continuous self-map of a manifold of dimension at least 2.*

Remark 3.4. *The idea of Yano was to show that a generic element of these spaces contains a horseshoe type structure.*

Theorem 3.5. [17] *A generic homeomorphism of S^d , $2 \leq d \leq \infty$, has infinite topological entropy.*

Theorem 3.6. [17] *A generic homeomorphism of the Cantor space has zero topological entropy.*

Theorem 3.7. [22] *A generic homeomorphism of a manifold of dimension d , $2 \leq d < \infty$, has the property that some power of it is semi-conjugate to the shift map and has infinite topological entropy.*

Theorem 3.8. [19] *A generic transitive homeomorphism of the Cantor space is conjugate to the universal adding machine and hence has zero topological entropy..*

Following the work of [1], [21], Daalderop and Fokkink [11] showed that

Theorem 3.9. *A generic measure-preserving homeomorphisms on a compact d -dimensional manifold, $d \geq 2$, is chaotic in the sense of Devaney.*

4 More on chaos on the Cantor Space

Recently, in a joint work with U.B. Darji [12], we have answered (see Theorem 4.5) the following natural

Query: what about Devaney chaos for a generic continuous map on the Cantor space?

In particular, we have shown that there is a dense subset of the space of all continuous self-maps of the Cantor space each element of which has infinite topological entropy and is Devaney chaotic on a subsystem and that, however, a generic continuous map of the Cantor space is neither Devaney chaotic nor Devaney chaotic on any subsystem and has zero topological entropy.

Here are the main results in [12]:

Proposition 4.1. [12] *The collection*

$$\mathcal{E} = \{f \in C(X) : \text{ent}(f) = +\infty\}$$

is dense.

Theorem 4.2. [12] *The collection $\mathcal{Z} = \{f \in C(X) : \text{ent}(f) = 0\}$ is a dense G_δ .*

For each $f \in C(X)$ and k , let $P_k(f) = \{x : f^k(x) = x\}$.

Proposition 4.3. [12] *Let $\mathcal{A}_k = \{f \in C(X) : P_k(f) \neq \emptyset\}$. Then, \mathcal{A}_k is closed. Moreover, in the case when X is the Cantor space, \mathcal{A}_k is nowhere dense.*

Corollary 4.4. [12] *A generic continuous self-map of the Cantor space has no periodic points.*

Theorem 4.5. [12] *A generic continuous self-map of the Cantor space has no periodic points and hence it is not Devaney chaotic on any subsystem.*

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