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ON THE ω-PROBLEM

Let $(X, \mathcal{T})$ be any T_1 topological space. Given a function $F\colon X \to \mathbb{R}$ and $x \in X$ we define an oscillation of f at x by $\omega(F, x) = \inf_U \sup_{x_1, x_2 \in U} |F(x_1) - F(x_2)|$, where the infimum is taken over all neighborhoods U of x. It is well-known that $\omega(F, \cdot)\colon X \to [0, \infty]$ is upper semi-continuous and vanishes at isolated points of X.

Let an upper semi-continuous function $f\colon X \to [0, \infty]$ vanishing at isolated points of X be given. If there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$ then we call F an ω-primitive for f. By the ω-problem on a topological space X we mean the problem of the existence of an ω-primitive for a given upper semi-continuous function vanishing at isolated point of X.

Complete solution of the ω-problem for metrizable spaces was obtained in 2001 [Ew2]. Namely, the following theorems are true:

Theorem 1. *[Ew2, Theorem 3] Let (X, d) be an arbitrary metric space and $f\colon X \to [0, \infty)$ an upper semi-continuous function which vanishes on $X \setminus X^d$. Then for each lower semi-continuous function $g\colon X \to (0, \infty)$ there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$ and $-g < F \leq f$.*

Theorem 2. *[Ew2, Theorem 4] Let (X, d) be an arbitrary metric space and $f\colon X \to [0, \infty]$ an upper semi-continuous function which vanishes on $X \setminus X^d$. Then there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$.*

To formulate equivalent condition for the existence of an ω-primitive for each continuous function $f\colon X \to \mathbb{R}$ we need a notion of a resolvable space, which was introduced in [H1].

Definition 1. *[H1] A topological space $(X, \mathcal{T})$ is said to be resolvable if it is dense in itself and contains two disjoint sets which are dense.*

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Theorem 3. *Let $(X, \mathcal{T})$ be any topological space dense in itself. The following conditions are equivalent:*

1. *X is resolvable,*
2. *for each continuous function $f\colon X \to [0, +\infty)$ there exists $f\colon X \to [0, +\infty)$ such that $\omega(F, \cdot) = f$,*
3. *there exist $\delta > 0$ and a continuous function $f\colon X \to [\delta, +\infty)$, which has an ω-primitive.*

Theorem 4. *Let $(X, \mathcal{T})$ be any topological space. For each $f\colon X \to [0, +\infty)$ the set $\{F\colon X \to \mathbb{R}\colon\ \omega(F, \cdot) = f\}$ is closed in the space $\mathbb{R}^X$ of all functions from X to $\mathbb{R}$ with the metric of uniform convergence*
$\operatorname{dist}(g, h) = \min\{1, \sup\{|g(x) - h(x)|\colon x \in X\}\}$.

Now, we give some sufficient conditions for the existence of an ω-primitive.

Theorem 5. *Let $(X, \mathcal{T})$ be a topological space dense in itself and let $f\colon X \to \mathbb{R}$ be any upper semi-continuous function. If a subset A of the space X satisfies the following conditions:*

1. $\operatorname{cl}(A) = \operatorname{cl}(X \setminus A) = X$,
2. *the set* $\{x \in A : f(x) - \limsup_{t \to x} f(t) > \varepsilon\}$ *is closed for each* $\varepsilon > 0$,
3. $\limsup_{(X \setminus A) \ni t \to x} f(t) = \limsup_{t \to x} f(t)$ *for each* $x \in A$,

then the function $F\colon X \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} f(x) & if \quad x \in X \setminus A, \\ -\big(f(x) - \limsup_{t \to x} f(t)\big) & if \quad x \in A, \end{cases}$$

is an ω-primitive for f.

Theorem 6. *Let $(X, \mathcal{T})$ be a regular first countable and separable topological space. Then for each pair of functions $f\colon X \to [0, +\infty)$ upper semi-continuous and $g\colon X \to (0, +\infty)$ lower semi-continuous, there exists a function $f\colon X \to [0, +\infty)$ such that $\omega(F, \cdot) = f$ and $-g < F \leq f$.*

In the theory of the ω-problem for metric spaces very important role plays the case of so called massive spaces. Each upper semi-continuous function f defined on a massive metric space has an ω-primitive of very easy form $F = f \cdot \chi_A$, where χ_A is a characteristic function of some set A and A is of type F_σ. We study problem of the existence of a set $A \subset X$ for which $\omega(f \cdot \chi_A, \cdot) = f$ for a function defined on a massive topological space.

Definition 2. *[Du] Let $(X, \mathcal{T})$ be a topological space. We say that X is:*

- *σ-discrete at a point $x \in X$, if there exists a neighborhood U of x which is a σ-discrete set,*
- *massive at a point $x \in X$, if it is not σ-discrete at x, this means that any neighborhood of x is not a σ-discrete set,*
- *massive, if it is massive at each point $x \in X$.*

Obviously, each massive topological space is dense in itself.

Definition 3. *We say that a topological space $(X, \mathcal{T})$ is weakly regularly resolvable if it contains a dense subset which is σ-discrete. We say that a subset $A \subset X$ is weakly regularly resolvable if there exists a σ-discrete set $B \subset A$ such that $A \subset \mathrm{cl}(B)$.*

Theorem 7. *Let $(X, \mathcal{T})$ be a massive topological space. The following conditions are equivalent.*

1. *each subset of X, which is intersection of two sets, one of which is open and the second one is closed, is weakly regularly resolvable,*
2. *for each upper semi-continuous function $f : X \to [0, +\infty)$ there exists a σ-discrete set $A \subset X$ such that $\omega(f \cdot \chi_A, \cdot) = f$.*

Theorem 8. *Let $(X, \mathcal{T})$ be a massive topological space such that each subset of X which is intersection of two sets, first one closed and second one open, is weakly regularly resolvable. Moreover, let $f : X \to [0, +\infty]$ be any upper semi-continuous function, $A_\infty = \{x \in X : f(x) = +\infty\}$ and put $\widetilde{f} = f \cdot \chi_{X \setminus A_\infty}$. The following condition are equivalent:*

1. *there exists a σ-discrete set $A \subset \{x \in X : f(x) < +\infty\}$ such that $\omega(f\chi_A, \cdot) = f$,*
2. *$A_\infty \subset \{x \in X : M_{\widetilde{f}}(x) = +\infty\}$,*
3. *if $f(x) = \infty$ then for each $K > 0$ and for each neighborhood U of x there exists $y \in U$ for which $K < f(y) < \infty$.*

References

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