Real Analysis Exchange
Summer Symposium 2011, pp. 124–128

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ON KANTOROVICH'S RESULT ON THE SYMMETRY OF DINI DERIVATIVES

Results presented in the talk were obtained in cooperation with L. Zajíček and can be found in entirety in our joint article [3].

For $f : (a, b) \to \mathbb{R}$, let A_f denote the set of all points at which f is Lipschitz from the left and is not Lipschitz from the right. Symmetrically, let B_f denote the set of all points at which f is Lipschitz from the right and is not Lipchitz from the left.

Note that the classical Denjoy-Young-Saks theorem (see e.g. [5, §70]) gives that A_f and B_f are always Lebesgue null.

The set A_f (in the case when f is continuous) was considered by L.V. Kantorovich in his early work [2]. His result [2, Theorem II, p. 161] reads as follows:

Theorem K. *If f is a continuous function on (a, b), then A_f is a (k_d)-reducible set.*

A set $E \subset \mathbb{R}$ is called (k_d)-*reducible*, if for each perfect set $F \subset E$ the set of all points $x \in F$ at which F is not strongly right porous is not dense in F. In other words, E is (k_d)-reducible if and only if each perfect set $F \subset E$ has a portion F^* (i.e. a nonempty set of the form $F^* = F \cap (c, d)$) which is a strongly right porous set.

Let us recall the definition of strong right porosity:

Definition. Let $E \subset \mathbb{R}$ and $x \in \mathbb{R}$.

(i) We say that E is *strongly right porous at x* if there exists a sequence of open intervals $\{I_k\}_{k=1}^{\infty}$ on the right from x such that $|I_k| \to 0$, $\operatorname{dist}(x, I_k) \to 0$, $\frac{|I_k|}{\operatorname{dist}(x,I_k)} \to \infty$ and $I_k \cap E = \emptyset$ for every $k \in \mathbb{N}$.

Mathematical Reviews subject classification: Primary: 26A27, 28A05
Key words: Dini derivative, one-sided Lipschitzness, σ-porous set, strong right porosity, abstract porosity, first keyword

(ii) We say that E is *strongly right porous* if E is strongly right porous at each point of E.

(iii) We say that E is *σ-strongly right porous* if it can be expressed as a countable union of strongly right porous sets.

For an alternative definition of unilateral strong porosity as well as for other definitions of similar porosity notions see e.g. [5] or [7].

Independently on [2], the sets A_f and B_f were studied in [6]. The result of [6, Theorem 2] states that $A_f \cup B_f$ is σ-strongly porous even for an arbitrary function f on (a, b) and the proof clearly gives that

$$A_f \text{ is a } \sigma\text{-strongly right porous set.} \tag{1}$$

For an alternative proof, where (1) can be seen more easily, see the proof of [5, Theorem 73.2].

Theorem K is (for a continuous function f) stronger than (1), since it easily implies that

($*$) *there exists a σ-strongly right porous set $A \subset (a, b)$ such that $A \subset A_f$ for no continuous function f on (a, b).*

Slightly changing the proof of (1) from [5], we obtain Theorem K, and so also ($*$), even for an arbitrary function f. Moreover, we fully characterize the smallness of sets A_f, proving the following result (see [3, Theorem 1.2]):

Theorem 1. *Let $a, b \in \mathbb{R}^*$ and $A \subset (a, b)$. The following statements are equivalent:*

(i) There exists a continuous function $f : (a, b) \to \mathbb{R}$ such that $A \subset A_f$.

(ii) There exists a function $f : (a, b) \to \mathbb{R}$ such that $A \subset A_f$.

(iii) There exists a sequence $\{A_n\}_{n=1}^{\infty}$ of strongly right porous sets such that $A_1 \subset A_2 \subset \ldots$ and $A = \bigcup_{n=1}^{\infty} A_n$.

The condition (iii) gives a full simple characterization of the hereditary class generated by the sets of the form A_f, i.e. the class

$$\mathcal{H} := \{A \subset (a, b) : \ A \subset A_f \text{ for some } f : (a, b) \to \mathbb{R}\}.$$

This characterization shows that $\mathcal{H}$ is strictly smaller than the class $\mathcal{S}$ of all σ-strongly right porous sets, since there exists a closed σ-strongly right porous set $S \subset \mathbb{R}$ such that $S \neq \bigcup_{n=1}^{\infty} S_n$ whenever $S_n \subset S_{n+1}$ and S_n is strongly right porous for every $n \in \mathbb{N}$ (see [3, Proposition 3.2]). On the other hand, Theorem 1 immediately implies that the σ-ideal generated by the sets of the form A_f coincides with $\mathcal{S}$.

For a continuous function $f:(a,b)\to\mathbb{R}$, denote

$$F_n := \left\{ x\in(a,b) : \left|\frac{f(y)-f(x)}{y-x}\right| \le n \text{ for each } y\in\left(x-\frac{1}{n},x\right)\right\} \quad (n\in\mathbb{N}).$$

It is easy to see that all F_n are closed and $F_n \nearrow L^-$, where L^- is the set of all points at which f is Lipschitz from the left. So L^- is an F_σ set. Similarly, the set L^+ of all points at which f is Lipschitz from the right is an F_σ set. Set $H := \mathbb{R}\setminus L^+$. Then H is a G_δ set. Since sets $F_n\cap H$ are strongly right porous for every $n\in\mathbb{N}$ and $A_f = \bigcup_{n=1}^\infty F_n\cap H$, we get that if

(i) $A = A_f$ for some continuous function $f:\mathbb{R}\to\mathbb{R}$,

then

(ii) there exist closed sets $F_1\subset F_2\subset ...$ and a G_δ set H such that each $F_n\cap H$ is strongly right porous and $A_f=\bigcup_{n=1}^\infty F_n\cap H$.

Problem 2. We do not know whether the implication (ii)$\Rightarrow$(i) holds. In case of positive answer, we would obtain a full simple characterization of sets of the form A_f for continuous functions f.

We also do not know the answer to the following (more general) question:

Question 3. Let $B\subset\mathbb{R}$ be a Borel (k_d)-reducible set. Do then exist strongly right porous sets A_n such that $A_1\subset A_2\subset ...$ and $\bigcup_{n=1}^\infty A_n = B$?

Finally, we present some easy but non-trivial observations on (k_d)-reducible sets and monotone unions of strongly right porous sets (see [3, Proposition 5.1] and [3, Proposition 5.2]). It is natural to work with an "abstract porosity P" instead of strong right porosity.

Let (X,ϱ) be a metric space. The open ball with center $x\in X$ and radius $r>0$ will be denoted by $B(x,r)$. By an *abstract porosity* on X we will mean a relation P between points of X and subsets of X (i.e. $P\subset X\times 2^X$) fulfilling the following "axioms":

(A1) If $A\subset B\subset X$, $x\in X$ and $P(x,B)$, then $P(x,A)$.

(A2) $P(x,A)$ if and only if there exists $r>0$ such that $P(x,A\cap B(x,r))$.

(A3) $P(x,A)$ if and only if $P(x,\overline{A})$.

We say that $A\subset X$ is

(i) *P-porous at $x \in X$*, if $P(x, A)$,

(ii) *P-porous* if $P(x, A)$ for every $x \in A$,

(iii) *σ-P-porous* if A is a countable union of P-porous sets.

It is easy to see that strong right porosity on $\mathbb{R}$ (as most of natural versions of porosity) is an abstract porosity which also fulfills

(A4) If $x \in X$ is not an isolated point of X, then $P(x, \{x\})$.

Proposition 4. *Let (X, ϱ) be a metric space, P an abstract porosity on X and $A \subset X$. Then the following conditions are equivalent:*

(i) $A_n \nearrow A$, where each A_n is P-porous.

(ii) $B_n \nearrow A$, where each B_n is P-porous and relatively closed in A.

(iii) $A = \bigcup_{n=1}^{\infty} C_n$, where each C_n is P-porous and relatively closed in A.

We say that $A \subset X$ is *P-reducible* if each nonempty closed set $F \subset A$ contains a P-porous subset with nonempty relative interior in F.

Proposition 5. *Let (X, ϱ) be a separable topologically complete metric space without isolated points, $A \subset X$ and let P be an abstract porosity on X satisfying (A4). Then the following conditions are equivalent:*

(i) A is P-reducible.

(ii) Each nonempty perfect set $F \subset A$ contains a P-porous set with nonempty relative interior in F.

(iii) Each closed set $F \subset A$ is a countable union of closed P-porous sets.

Hence we obtain the following result:

Corollary 6. *Let (X, ϱ) be a separable topologically complete metric space without isolated points, $A \subset X$ and let P be an abstract porosity on X satisfying (A4).*

(a) If there exist P-porous sets P_n with $P_n \nearrow A$, then A is P-reducible.

(b) If A is closed and P-reducible, then there exist P-porous sets P_n with $P_n \nearrow A$.

If $X = \mathbb{R}$ and P is strong right porosity, then Corollary 6 (b) does not hold for an arbitrary $A \subset X$ (and it seems that it also does not hold for all other interesting cases). Indeed, if $A \subset \mathbb{R}$ is a Bernstein set of the second category (see [4, Theorem 5.4]), then A is (k_d)-reducible (since it contains no perfect subset), but A is not a σ-strongly right porous set. This fact motivates the following natural question:

Question 7. Does Corollary 6 (b) hold for Borel sets $A \subset X$?

The answer to this question is negative in some non-trivial cases and positive in others (see [3, Remark 5.4 (2)] for examples).

If $X = \mathbb{R}$ and P is strong right porosity, then the question coincides with Question 3. We also do not know the answer e.g. in the case when $X = \mathbb{R}$ (or $X = \mathbb{R}^n$) and P is the upper porosity on X.

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