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THE SHARKOVSKIĬ THEOREM FOR SPACES OF MEASURABLE FUNCTIONS

The classical Sharkovskiĭ Theorem classifies continuous real functions according to iteration periods of periodic points which they possess. This classification is based on the so called "Sharkovskiĭ ordering" on natural numbers defined as follows:

$$\begin{array}{ccccccccc} 3 & \rhd & 5 & \rhd & 7 & \rhd & 9 & \rhd & \dots \\ 2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & 2\cdot 9 & \rhd & \dots \\ 2^2\cdot 3 & \rhd & 2^2\cdot 5 & \rhd & 2^2\cdot 7 & \rhd & 2^2\cdot 9 & \rhd & \dots \\ & & & & \vdots & & & & \\ \dots & \rhd & 2^3 & \rhd & 2^2 & \rhd & 2 & \rhd & 1 \end{array}$$

Theorem (Sharkovskiĭ). *If a continuous function $f\colon [0,1] \to [0,1]$ $(\mathbb{R} \to \mathbb{R})$ has a point of period n, then it has points of period k for any $k \lhd n$.*

The theorem can be generalized in numerous directions. We consider the so called random generalizations. There are few generalizations of this kind, the Random Sharkovskiĭ Theorem from paper [2] by J. Andres is one of them. It has an assumption of completeness of the underlying measurable space. We omit that assumption formulating the Measurable Sharkovskiĭ Theorem.

Assume that (Ω, Σ) is a measurable space, $\mathcal{I}$ — a σ-ideal on Ω such that $\Omega \notin \mathcal{I}$, $X = \mathbb{R}$ or $X = [0,1]$, $\mathfrak{M}$ — space of measurable functions $f\colon \Omega \to X$, $\mathcal{B}(X)$ — Borel sets. A function $f\colon \Omega \times X \to X$ is called a measurable operator if it is measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X)$. A measurable operator f is called continuous if $f(\omega, \cdot)$ is continuous for each $\omega \in \Omega$. A sequence of

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measurable functions $(\xi_i\colon \Omega \to X)_{i=1}^k$ is called a $(\Sigma, \mathcal{I})$-measurable k-orbit of the operator f, if

$$\forall_{i\leq k-1}\, \xi_{i+1}(\omega) = f(\omega, \xi_i(\omega)), \xi_1(\omega) = f(\omega, \xi_k(\omega))$$

for almost all $\omega \in \Omega$ (all apart from a set in $\mathcal{I}$) and there is no $m < k$, $m \,|\, k$, such that for almost all $\omega \in \Omega$

$$\forall_{s\leq \frac{k}{m}} \forall_{i\leq m}\, \xi_{(s-1)m+i}(\omega) = \xi_i(\omega).$$

These definitions coincide with the definitions of random operator and random orbit if we take Lebesgue measurable sets as a measurable space and null sets as a σ-ideal.

Our main result from paper [1] is:

Theorem 1 (Measurable Sharkovskiĭ Theorem)**.** *Let $f\colon \Omega \times X \to X$ be a continuous measurable operator. Then if f has a measurable n-orbit, then it has a measurable k-orbit for each $k \lhd n$.*

It can be reformulated such that it concerns operators on spaces of measurable functions (actually spaces of classes of equivalence of measurable functions with the relation of equivalence of equality almost everywhere).

Corollary 2. *Let the function $\mathfrak{f}\colon \mathfrak{M} \to \mathfrak{M}$ be given by the formula $\mathfrak{f}(\xi)(\omega) = f(\omega, \xi(\omega))$ for almost every $\omega \in \Omega$ and for all $[\xi] \in \mathfrak{M}$, where $f\colon \Omega \times X \to X$ is a continuous measurable operator. Then if $\mathfrak{f}$ has an n-orbit, then it has a k-orbit for each $k \lhd n$.*

To get the generalization we use the method of transformation to deterministic case involving the Kuratowski-Ryll-Nardzewski selection theorem instead of the Aumann-von Neumann selection theorem, used in paper by J. Andres. This forces us to use some more sophisticated facts about measurable multifunctions.

We also state a version of the theorem where the measurable operator is a multifunction and a version where the orbits belong to the L^p-space.

References

[1] Paweł Barbarski, *The Sharkovskiĭ Theorem for spaces of measurable functions*, J. Math. Anal. Appl. **373** (2011), no. 2, 414–421.

[2] Jan Andres, *Randomization of Sharkovskii-type theorems*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1385–1395.