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A PLETHORA OF HIGH DIMENSIONAL ATTRACTORS

1 Introduction

In 1989, Bruckner, Agronsky, Ceder and Pearson characterized the attractors of continuous interval maps as being either finite unions of disjoint closed intervals or nowhere dense closed sets. Here we investigate attractors of self maps of the unit cube in $\mathbb{R}^n$.

2 The One Dimensional Story

Let $I = [0, 1]$ and let $I^n = I^n$. If $f : I^n \to I^n$ is a continuous function and $x_0 \in I^n$, then the *orbit* of x_0 under f, which we denote by $\mathrm{orb}_f(x_0)$, is the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ where for any $x \in I^n$ we define $f^0(x) = x$ and $f^{n+1}(x) = f^n(f(x))$. The *attractor* or *ω-limit set* of f at x_0, which we denote by $\omega_f(x_0)$, is the set of subsequential limit points of $\mathrm{orb}_f(x_0)$. Since I^n is compact, $\omega_f(x_0)$ always contains at least one point.

We are interested in identifying those subsets of I^n that are attractors of some continuous function $f : I^n \to I^n$. In 1989, Agronsky, Bruckner, Ceder, and Pearson, [1] characterized the attractors of continuous functions $f : I \to I$ as follows.

Key words: attractor, omega limit set

*Presenter

Theorem 1 (Agronsky Bruckner, Ceder, Pearson, *RAE* (1989)). *A compact set A is an attractor iff*

a. A is zero dimensional (empty interior)

b. A is the finite disjoint union of closed intervals

Here we examine extensions of Agronsky, Bruckner, Ceder, and Pearson's results to continuous functions $f : I^n \to I^n$ and look at some specific examples. In the main we restrict our focus to the n-cube I^n, but on occasion will need more general topological results. We'll also use the following one dimensional theorem.

Theorem 2 (Blokh, Bruckner, Humke, Smital *TAMS* (1994)). *If $f : I \to I$ is continuous, then A is an attractor of f iff f locally expands A at each of its points.*

3 The Two (+) Dimensional Case

The following are easy to see from the one dimensional case.

- If A is compact and zero dimensional, then A is an attractor.
- If A is a finite union of disjoint closed 2-cells, then A is an attractor
- If A is a finite union of disjoint closed 1-cells, then A is an attractor
- If A is the disjoint union of an arc and a 2-cell then A is an attractor

The first real two dimensional result belongs to Agronsky and Ceder in 1996 when they prove the following theorem:

Theorem 3 (Agronsky,Ceder *RAE* (1996)). *If A is the disjoint union of Peano continua, then A is an attractor.*

And a continuum is Peano iff it is locally connected, so in light of this we focus on the following question.

Question 1. *Which continua are attractors?*

The following are easy to see.

1. Every 1 dimensional continuum is a becalmed attractor (that is, an attractor of a function which is the identity map when restricted to the attractor).

2. Some continua are not attractors (for example, a ray spun down onto a disc).

We provide several criteria under which a disc unioned with a one dimensional continuum is an attractor.

References

[1] S. J. Agronsky, A.M. Bruckner, J.G. Ceder and T.L. Pearson, *The Structure of ω-limit Sets for Continuous Functions* Real Anal. Exchange, **15(2)**, (1989-1990), 483–510.

[2] A. Blokh, A. M. Bruckner, P.D. Humke and J. Smítal, *The Space of ω-Limit Sets of a Continuous Map of the Interval*, Trans. Amer. Math. Soc., **348(4)**, (1996), 1357–1372.