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CONVEX FUNCTIONS AND PEANO DERIVATIVES

Abstract

This talk is on my two recent papers. The first paper deals with Peano differentiable functions of several variables. Let f be a function of several variables that is n times Peano differentiable. Andreas Fischer proved that if there is a number M such that for each α , with $|\alpha| = n$; all Peano partials are bounded from below ($f_\alpha \geq M$) or all are bounded from above ($f_\alpha \leq M$) then f is n times differentiable in the usual sense. Here that result is improved to permit the type of one-sided boundedness to depend on α . A nice result on equality of mixed partial derivatives is also obtained. The second one deals with differentiability properties of n -convex functions. The main result of this paper is that if f is n -convex on a measurable subset E of \mathbb{R} , then f is $(n-2)$ times differentiable, $(n-2)$ times Peano differentiable and the corresponding derivatives are equal everywhere on E . Moreover $f^{(n-1)} = f_{(n-1)}$ except for a countable set, and $f_{(n-1)}$ is approximately differentiable with approximate derivative equal to the n^{th} approximate Peano derivative of f almost everywhere.

In this talk I will be covering some of my recent results on Peano differentiable functions in several variables, as well as on differentiability properties of n convex functions in one variable. These results are my joint work with Cliff Weil [3] and in the case of n convex functions with Cliff Weil and Ralph Svetic [2].

First let me remind you of a definition of Peano derivatives. Often authors will assume that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is n times differentiable at a point x in order to use that the value of the function can be approximated close to x by a polynomial of degree n . It seems reasonable that if such an approximation is

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desired, then we can simply assume that one exists, and observe that it does exist if the differentiability condition is assumed. In 1891 G. Peano. (See [7].) first introduced this idea of approximating a function by a polynomial, which later became known as Peano differentiation.

To state the definition of Peano differentiation in several variables, we first recall the standard notation used when working with partial derivatives of functions of several variables. Let $d \in \mathbb{N}$ and let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$ be a d -tuple of non negative integers. Then $\boldsymbol{\alpha}! = \alpha_1! \cdot \alpha_2! \cdot \dots \cdot \alpha_d!$, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \dots + \alpha_d$ and if $\mathbf{x} \in \mathbb{R}^d$, then $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \dots \cdot x_d^{\alpha_d}$ with the usual conventions that $0! = 1$ and $0^0 = 1$. Also for $p = 1, 2, \dots, d$ we use $\partial_p f$ to denote the partial derivative of f with respect to the p^{th} variable. Moreover for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\partial_{\boldsymbol{\alpha}} f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_d^{\alpha_d} f$ (where $\partial_p^k f$ means the partial with respect to the p^{th} variable taken k times). This notation presupposes the equality of the mixed partial derivatives.

Definition 1. Let $n \in \mathbb{N}$, let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and let $\mathbf{x} \in \mathbb{R}^d$. Then f is n times Peano differentiable at \mathbf{x} means for each index $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \leq n$ there is a number $f_{(\boldsymbol{\alpha})}(\mathbf{x})$ such that

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \sum_{1 \leq |\boldsymbol{\alpha}| \leq n} \frac{f_{(\boldsymbol{\alpha})}(\mathbf{x})}{\boldsymbol{\alpha}!} \mathbf{h}^\alpha + o(\|\mathbf{h}\|^n).$$

As in the one variable case, if f is n times Peano differentiable, then the coefficients, $f_{(\boldsymbol{\alpha})}$ called Peano partials, are unique. Note that Peano differentiability implies that f is differentiable at \mathbf{x} and that the first order partials satisfy $\partial_p f(\mathbf{x}) = f_{(\mathbf{p})}(\mathbf{x})$ for each $p = 1, 2, \dots, d$ where \mathbf{p} denotes the d -tuple having 1 in the p^{th} position and 0 elsewhere.

An extensive study of the Peano derivative in one variable, was done by H. W. Oliver in 1954. (See [6].) He established that if a function, f is n times Peano differentiable on an interval, then the function $f_{(n)}$ has many of the properties known for an ordinary derivative. In addition he proved that if $f_{(n)}$ is bounded above or below on the interval, then it is an ordinary derivative; i.e., $f_{(n)} = f^{(n)}$.

The goal of this paper is to prove an analogue of Oliver's boundedness result for Peano derivatives in several variables. In 2008 (See [5].) Andreas Fischer proved a result of this type for $n = 2$ while for $n \geq 3$ he assumed that all of the n^{th} order Peano partials are bounded above or all bounded below. Here for $n \geq 3$ we allow the possibility that for $|\boldsymbol{\alpha}| = n$, some $f_{(\boldsymbol{\alpha})}$ are bounded from above while the others are bounded from below. Our main result is

Theorem 1. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be n times Peano differentiable on \mathbb{R}^d and for each d -tuple, α , with $|\alpha| = n$ the n^{th} order Peano partial $f_{(\alpha)}$ is bounded from above or from below on \mathbb{R}^d . Then f is n times differentiable in the usual sense on \mathbb{R}^d ; that is, all $(n - 1)$ -st partials of f exist and they are differentiable. Moreover the values of the partials are equal to the corresponding values of Peano partials; that is, if a partial of f is obtained by taking γ_p partials with respect to x_p for $p = 1, 2, \dots, d$ in some order, then this partial is equal to the Peano partial $f_{(\gamma)}$. In particular all the mixed partials are equal.*

Most papers that deal with higher order differentiation in several variables assume that the functions under study are C^n in order to conclude that the mixed partial derivatives are equal. However as was shown by W. H. Young in 1908 (See [8].) the equality of the mixed partial derivatives can be deduced by assuming only that the partial derivatives of order $n - 1$ are differentiable functions. Additional conditions that imply equality of the mixed partial derivatives can be found in the main result. A special case of it is the following result.

Corollary 2. *Let $U \subset \mathbb{R}^d$ be open and let $f : U \rightarrow \mathbb{R}$ have second order partials $\partial_{pp}^2 f$ and $\partial_{qq}^2 f$ on U . Assume that $\partial_{pp}^2 f$ is bounded from below or from above on U and the same for $\partial_{qq}^2 f$. If f is twice Peano differentiable at $\mathbf{w} \in U$, then $\partial_{pq}^2 f(\mathbf{w})$ and $\partial_{qp}^2 f(\mathbf{w})$ exist and equal.*

In the above assertion the assumption that f is twice Peano differentiable can be replaced by f is twice differentiable in the usual sense, because the second assumption implies the first. This version of the result is substantially better than the usual assumption that the second order partials exist and are continuous.

My second part of the talk is on differentiability properties of n convex functions defined on a measurable subset, E of \mathbb{R} . We will consider approximate derivatives, which are defined by replacing $\lim_{h \rightarrow 0}$ from the definition of ordinary ones with $\lim_{h \rightarrow 0, h \in D}$ where D has density 1 at 0. The n -th divided difference $[f; V]$, where $V = \{x_0, x_1, \dots, x_n\}$ is defined by

$$[f; V] = \sum_{i=0}^n \frac{f(x_i)}{\prod_{\ell \neq i} (x_i - x_\ell)}.$$

Definition 2. Let $f : E \rightarrow \mathbb{R}$ and let $n \in \mathbb{N}$. Then f is n -convex on E means for each $V \subset E$ of cardinality $n + 1$, $[f; V] \geq 0$.

Consequently f is 0-convex on E means $f \geq 0$ on E , 1-convex on E means f is increasing on E and 2-convex means f is convex in the usual sense. It

is easy to show that a polynomial of degree n is n -convex if and only if the leading coefficient of f is positive. If f is n -convex on a set, E , then, (as we've shown in this paper), f has a unique n -convex extension to the closure of E . Consequently, it could be assumed that E is closed, but there is no advantage to doing so.

The motivation for this paper is the article by P. S. Bullen and S. N. Mukhopadhyay ([1]) in which the following is asserted.

Theorem 3. *If f is n -convex on a measurable set $E \subset [a, b]$ on which $(n-1)$ -th Peano derivative, $f_{(n-1)}$ exists finitely, then both n -th approximate Peano, $f_{(n),ap}$ of f and approximate derivative $(f_{(n-1)})'_{ap}$ of f_{n-1} exist finitely and are equal almost everywhere in E .*

In 2000, M. Laczko (see [4]) pointed out that the proof is not valid, but didn't determine if the assertion itself was true or false. In this paper we show that the assertion is true. In fact we show a little bit more.

Theorem 4. *Let $n \in \mathbb{N}$ and let $f : E \rightarrow \mathbb{R}$ be n -convex. Then f is n times approximately Peano differentiable almost everywhere on E . Moreover $f_{(n-1)}$ is approximately differentiable with $(f_{(n-1)})'_{ap} = f_{(n),ap}$.*

One of our other results generalizes well known result for convex functions on intervals. Here $E^+ \cap E^-$ denotes the set of points that is both right and left accumulation point of E .

Theorem 5. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $f : E \rightarrow \mathbb{R}$ be n -convex. Then $f^{(i)}(u) = f_{(i)}(u)$ for $u \in E^+ \cap E^-$ and $i = 1, \dots, n-2$. Also except for countably many $u \in E$, $f^{(n-1)}(u) = f_{(n-1)}(u)$.*

References

- [1] P. S. Bullen and S. N. Mukhopadhyay, *Relations Between Some Generalized n -th-order Derivatives*, Fund. Math., **85** (1974), 257–276.
- [2] H. Fejzić, R. Svetić, C. Weil, *Differentiation of n -convex functions*, Fund. Math., **209 no. 1** (2010), 9–25.
- [3] H. Fejzić, C. Weil, *A property of Peano derivatives in several variables*, Proc. Amer. Math. Soc., (to appear)
- [4] M. Laczko, *Infinite Peano Derivatives*, Real Anal. Exchange, **26** (2000/2001), 811–825 .

- [5] Andreas Fischer, *Differentiability of Peano derivatives*, Proc. Amer. Math. Soc., **136** (2008), 1779–1785.
- [6] H. W. Oliver, *The Exact Peano Derivative*, Trans. Amer. Math. Soc., **76** (1954), 444–456.
- [7] G. Peano, *Sulla formula di Taylor*, Atti Accad. sci. Torino, **27** 1891-2, 40–46.
- [8] W. H. Young, *On differentials*, Proc. London Math. Soc., **7** 1908, 157–180.