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RECENT PROGRESS IN THE THEORY OF MONOTONE METRIC SPACES

Abstract

A metric space (X, d) is called *monotone* if there is a linear order $<$ on X and $c > 0$ such that $d(x, y) \leqslant c\,d(x, z)$ for all $x < y < z$ in X. A brief account of recent results in the theory of monotone spaces is presented.

1 Monotone spaces

Let $c > 0$. A metric space (X, d) is called *c-monotone* if there is a linear order $<$ on X such that $d(x, y) \leqslant c\,d(x, z)$ for all $x < y < z$ in X. Say that (X, d) is *monotone* if it is c-monotone for some $c > 0$.

This notion was conceived in [12] to aid a problem regarding Hausdorff dimension. Since then a number of papers focusing on monotone metric spaces were written, some of them published, others to appear, and several more papers are under preparation, e.g. [6, 5, 2, 10, 13, 7, 3, 8] A brief review of the state of the theory of monotone metric spaces appeared two years ago in [11]. The present paper gives a concise account of several recent papers [13, 7, 3, 8] not covered by the review [11].

2 Functions with monotone graphs

My talk at the Real Analysis Symposium in Wooster 2010 initiated a discussion of several participants on the differentiability properties of continuous functions whose graphs ar monotone subsets of the plane. The discussion led

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to a paper [13] authored by O. Zindulka, M. Hrušák, T. Matrai, A. Nekvinda, and V. Vlasák. All results of this section come from this paper.

Consider a continuous function $f : I \to \mathbb{R}$ on an interval I. Denote its graph by $\mathsf{G}(f)$ and let $\psi_f : x \mapsto (x, f(x))$ be the natural parametrization of $\mathsf{G}(f)$. It is easy to see that if the graph $\mathsf{G}(f)$ is a monotone subset of the plane, then the natural order on $\mathsf{G}(f)$ defined by $\psi_f(x) < \psi_f(y)$ iff $x < y$ witnesses monotonicity of $\mathsf{G}(f)$. This uniqueness of the witnessing order makes investigation of monotone graphs easier.

It is not hard to see that if f is differentiable at every point, then $\mathsf{G}(f)$ is a countable union of monotone sets. At first sight it seems that differentiability of f and the monotonicity of $\mathsf{G}(f)$ may be related even closer. The original hope was that if the graph is monotone, then f is differentiable at a substantial portion of I. It however turned out that the situation is much more delicate.

Let us first have a look at the pointwise monotonicity features of the graph. We need to introduce some notions and notation.

Definition 2.1. • Let $f : I \to \mathbb{R}$ be a continuous function and $c \geqslant 1$. A point $y \in I$ is an $\mathcal{M}$-*point* of f if there is $c > 0$ and $\varepsilon > 0$ such that

$$\text{for all } x \in (y-\varepsilon),\ z \in (y+\varepsilon) \quad |\psi(x) - \psi(y)| \leqslant c|\psi(x) - \psi(z)|. \tag{1}$$

• The set of all $\mathcal{M}$-points of f is denoted $\mathcal{M}(f)$. The subset of the graph $\{\psi_f(x) : x \in \mathcal{M}(f)\}$ is denoted $\textbf{Mon}(f)$.

Say that a metric space is σ-monotone if it is a countable union of monotone subspaces. It is not difficult to prove that $\textbf{Mon}(f)$ is σ-monotone. In particular, if all points of i are $\mathcal{M}$-points, then $\mathsf{G}(f)$ is σ-monotone. In the other direction, if $\mathsf{G}(f)$ is σ-monotone, then $\operatorname{int} \mathcal{M}(f)$ is dense in I, i.e. $\textbf{Mon}(\mathsf{G}(f))$ contains an open dense subset of $\mathsf{G}(f)$. Moreover, using Baire category argument one can prove a profound connection between $\mathcal{M}$-points and monotone subsets of $\textbf{Mon}(f)$: Every monotone set $M \subseteq \mathsf{G}(f)$ is nowhere dense in $\mathsf{G}(f)$ if and only if $\operatorname{int} \mathcal{M}(f) = \emptyset$.

Differentiability vs. pointwise monotonicity

Let us have a look at Dini derivatives at $\mathcal{M}$-points. The four Dini derivatives of f at x are denoted $\overline{f}^+(x)$, $\underline{f}^+(x)$, $\overline{f}^-(x)$ and $\underline{f}^-(x)$. If the four Dini derivatives at x equal, the common value is of course the derivative $f'(x)$. If the two right Dini derivatives at x are equal, the common value is called the *right derivative* and denoted $f^+(x)$; and likewise for the left side. The set of points where the derivative of f exists (infinite values are allowed) is denoted $\mathcal{D}(f)$.

Recall that a point $x \in I$ is called a *knot point of* f if $\overline{f}^-(x) = \overline{f}^+(x) = \infty$ and $\underline{f}^-(x) = \underline{f}^+(x) = -\infty$. The set of knot points of f is denoted $\mathcal{K}(f)$.

We also consider approximate Dini derivatives $\overline{f}^+_{\mathsf{app}}(x)$, $\underline{f}^+_{\mathsf{app}}(x)$, $\overline{f}^-_{\mathsf{app}}(x)$ and $\underline{f}^-_{\mathsf{app}}(x)$ and the *approximate derivative* $f'_{\mathsf{app}}(x)$ and *right* and *left approximate derivatives*. The set of points where the approximate derivative of f exists is denoted $\mathcal{D}_{\mathsf{app}}(f)$. Approximate knot points are defined in the obvious way. The set of approximate knot points of f is denoted $\mathcal{K}_{\mathsf{app}}(f)$.

Linear measure, i.e. 1-dimensional Hausdorff measure in $\mathbb{R}^2$ is denoted by $\mathscr{H}^1$.

Application of the ultimate version [1] of the Denjoy–Khintchine Theorem to $\mathcal{M}$-points shows that almost every $\mathcal{M}$-point is either a point of approximate differentiability or a knot point.

Theorem 2.2. *If $f : I \to \mathbb{R}$ is continuous, then*

(i) $\mathcal{D}(f) \subseteq \mathcal{M}(f)$,

(ii) *there is a set $E \subseteq I$ such that $\mathscr{H}^1(\mathsf{G}(f)|E) = 0$ and $\mathcal{M}(f) \subseteq \mathcal{D}_{\mathsf{app}}(f) \cup \mathcal{K}_{\mathsf{app}}(f) \cup E$. In particular, almost every $\mathcal{M}$-point $x \notin \mathcal{D}_{\mathsf{app}}(f)$ is a knot point.*

A careful calculation of Hausdorff measure of the set of knot points that satisfy (1) yields the following corollaries to the theorem.

Theorem 2.3. *If $f : I \to \mathbb{R}$ is continuous, then $\mathscr{H}^1(\mathbf{Mon}(f))$ is σ-finite. In particular,* $\dim_{\mathsf{H}} \mathbf{Mon}(f) = 1$.

Corollary 2.4. *If $f : I \to \mathbb{R}$ is continuous with a monotone graph, then $\mathscr{H}^1(\mathsf{G}(f))$ is σ-finite. In particular,* $\dim_{\mathsf{H}} \mathsf{G}(f) = 1$.

Functions with a monotone or σ-monotone graph

Let us now consider functions with a monotone graph. It turns out that for such a function derivatives and approximate derivatives coincide:

Proposition 2.5. *If $f : I \to \mathbb{R}$ is continuous with a monotone graph, then $\overline{f}^+_{\mathsf{app}}(x) = \overline{f}^+(x)$ for all $x \in I$. A similar statement holds for all Dini derivatives.*

Thus the above theorems yield:

Corollary 2.6. *If $f : I \to \mathbb{R}$ is continuous function with a monotone graph, then there is a set $E \subseteq I$ such that $\mathscr{H}^1(\mathsf{G}(f|E)) = 0$ and $I = \mathcal{D}(f) \cup \mathcal{K}(f) \cup E$. In particular, almost all points $x \notin \mathcal{D}(f)$ are knot points.*

Theorem 2.7. *If $f : I \to \mathbb{R}$ is a continuous function with a σ-monotone graph, then f is differentiable at a perfectly dense set, i.e. a set whose intersection with every nonempty interval contains a perfect set.*

Corollary 2.8. *If $f : I \to \mathbb{R}$ is a continuous function, then* $\operatorname{int} \mathcal{M}(f) \subseteq \overline{\mathcal{D}(f)}$.

One would hope for better results, e.g. that a function with a monotone graph be differentable almost everywhere. But it is not the case.

Theorem 2.9. *For every $c > 1$ there is a continuous, almost nowhere differentiable function $f : [0, 1] \to \mathbb{R}$ with a c-monotone graph.*

Note that it follows from the above results that such a function necessarily have the following properties:

- Every point of $[0, 1]$ is an $\mathcal{M}$-point,
- the function is almost nowhere approximately differentiable,
- almost all points are knot points (actually approximate knot points),
- the function has a derivative at a perfectly dense set.

$\mathcal{M}_1$-points and 1-monotone graphs

However, if the graph is 1-monotone, one can prove much more. An $\mathcal{M}$-point is termed an $\mathcal{M}_1$-point, if it satisfies (1) of Definition 2.1 with $c = 1$. It turns out that being an $\mathcal{M}_1$-point is almost everywhere equivalent to differentiability.

Theorem 2.10. *If $f : I \to \mathbb{R}$ is continuous, then there is a set $E \subseteq I$ such that $\mathscr{H}^1(\mathsf{G}(f|E)) = 0$ and $\mathcal{D}(f) \subseteq \mathcal{M}_1(f) \subseteq \mathcal{D}(f) \cup E$. In particular, f is differentiable at almost every $\mathcal{M}_1$-point.*

One of the first questions regarding monotone graphs was whether a monotone graph guarantees bounded variation. As we shall see later, it is not so, but the conclusion holds for 1-monotone graphs:

Theorem 2.11. *If I is compact and $f : I \to \mathbb{R}$ is continuous with a 1-monotone graph, then f is of bounded variation.*

Corollary 2.12. *If I is compact and $f : I \to \mathbb{R}$ is continuous, then f is of bounded variation if and only if it is a sum of two continuous functions with 1-monotone graphs.*

3 1-monotone curves and sets

Motivated by the paper [13], A. Nekvinda, D. Pokorný and V. Vlasák wrote a paper [7]. The questions they consider are:

(i) Suppose that I is compact and $f : I \to \mathbb{R}$ has a finite derivative at every point and a monotone graph. Does it follow that f is of bounded variation?

(ii) Suppose $M \subseteq \mathbb{R}^n$ is 1-monotone. Is it true that $\dim_{\mathsf{H}} M \leqslant 1$?

As to the first question, a rather involved construction gives a negative answer:

Theorem 3.1. *For every $c > 1$ there is a continuous function $f : [0,1] \to \mathbb{R}$ such that*

- *f is infinitely differentiable at every $x \in (0,1]$,*
- *$f'(0) = 0$,*
- *f is not of bounded variation,*
- *f has a c-monotone graph.*

The second question is answered by the following theorem, with a highly nontrivial proof.

Theorem 3.2. *If $M \subseteq \mathbb{R}^n$ is a monotone, bounded set, then $\mathscr{H}^1(M) < \infty$. Therefore*

- *every monotone curve in $\mathbb{R}^n$ is of bounded variation,*
- *every monotone set in $\mathbb{R}^n$ is of σ-finite linear measure.*

4 Lipschitz mappings onto cubes and Urbański conjecture

Another paper [3] by T. Keleti, A. Máthé and O. Zindulka does not concentrate on investigation of monotone spaces but rather an application.

In a recent paper [4], Mendel and Naor proved that every analytic metric space contains sets of nearly the same Hausdorff dimension that are Lipschitz equivalent to ultrametric spaces. In more detail:

Theorem 4.1 ([4])**.** *Let X be an analytic metric space. For every $\varepsilon > 0$ there is a compact set $Y \subseteq X$ such that $\dim_{\mathsf{H}} Y \geqslant \dim_{\mathsf{H}} X - \varepsilon$ and a bi-Lipschitz mapping $f : Y \to Z$ onto an ultrametric space.*

Keleti, Máthé and Zindulka noticed that since every ultrametric space is monotone and monotonicity is invariant under bi-Lipschitz mappings, the Mendel–Naor theorem yields

Theorem 4.2. *Let X be an analytic metric space. For every $\varepsilon > 0$ there is a compact monotone set $Y \subseteq X$ such that $\dim_{\mathsf{H}} Y \geqslant \dim_{\mathsf{H}} X - \varepsilon$.*

Since monotone spaces are, in a sense, rather simple, this trivial consequence of Mendel–Naor theorem is likely to find applications. Keleti, Máthé and Zindulka proved that monotone spaces nicely map onto cubes:

Theorem 4.3. *Let X be a compact monotone metric space and let k be a positive integer. Then X can be mapped onto the k-dimensional cube $[0,1]^k$ by a Lipschitz map if and only if X has positive k-dimensional Hausdorff measure.*

Combining the two theorems they obtained a general theorem on mapping analytic spaces onto cubes by Lipschitz maps:

Theorem 4.4. *Let X be an analytic metric space and let k be a positive integer. If $\dim_{\mathsf{H}} X > k$ then X can be mapped onto the k-dimensional cube $[0,1]^k$ by a Lipschitz map.*

This theorem can be used to solve a conjecture of Urbański. In [9] Urbański introduced the *transfinite Hausdorff dimension* of a metric space X:

$$\mathrm{tHD}(X) = \sup\{\mathrm{ind}\, f(Y) : Y \subset X,\ f : Y \to Z \text{ Lipschitz, } Z \text{ a metric space}\},$$

where ind denotes the transfinite small inductive topological dimension. He showed that if X is a metric space with finite Hausdorff dimension, then $\mathrm{tHD}(X) \leq \lfloor \dim_{\mathsf{H}} X \rfloor$, where $\lfloor . \rfloor$ denotes the floor function, and conjectured that if X is a metric space with finite Hausdorff dimension then either $\mathrm{tHD}(X) = \lfloor \dim_{\mathsf{H}} X \rfloor - 1$ or $\mathrm{tHD}(X) = \lfloor \dim_{\mathsf{H}} X \rfloor$.

It is not hard to see that this conjecture fails in general: It is with ZFC that there exist a set $X \subseteq \mathbb{R}^2$ of positive (outer) Lebesque measure and of cardinality less that continuum. For such a set $\dim_{\mathsf{H}} X = 2$ but $\mathrm{tHD}(X) = 0$. However, Keleti, Máthé and Zindulka noticed that Theorem 4.4 yields Urbański conjecture for analytic spaces:

Theorem 4.5. *Let X be an analytic metric space.*

- *If $\dim_{\mathsf{H}} A$ is finite but not an integer, then $\mathrm{tHD}(A) = \lfloor \dim_{\mathsf{H}} A \rfloor$,*
- *if $\dim_{\mathsf{H}} A$ is an integer, then $\mathrm{tHD}(A)$ is $\dim_{\mathsf{H}} A$ or $\dim_{\mathsf{H}} A - 1$,*
- *if $\dim_{\mathsf{H}} A = \infty$, then $\mathrm{tHD}(A) \geq \omega_0$.*

5 Set-theoretic line of research

M. Hrušák and O. Zindulka [2] studied cardinal invariants of the ideal **Mon** of σ-monotone subsets of the plane. Since this area is somewhat out of scope of this journal, we outline their results very briefly. We use the common set-theoretic notation.

First it is proved that $\mathbf{add}(\mathsf{Mon}) = \omega_1$ and $\mathbf{cof}(\mathsf{Mon}) = \mathfrak{c}$. The other two invariants are more involved: $\mathbf{non}(\mathsf{Mon}) \geqslant \mathfrak{m}_{\sigma\text{-linked}}$, but $\mathbf{non}(\mathsf{Mon}) \geqslant \mathfrak{m}_{\sigma\text{-centered}}$ is consistent. Also, $\mathbf{cov}(\mathsf{Mon}) < \mathfrak{c}$ and $\mathbf{cov}(\mathsf{Mon}) > \mathbf{cof}(\mathcal{N})$ are consistent. In order to get these result the authors use lower porous sets and, as a by-product, they also obtain a number of results regarding cardinal invariants of the ideals of σ-lower porous sets in Euclidean spaces.

Further results in this direction appear in a PhD thesis [8] of Arturo Antonio Martínez Celis Rodríguez.

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