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DIFFERENTIABLE EXTENSIONS FROM SPECIAL CLOSED SUBSETS OF $\mathbb{R}^n$

Results presented in the talk were obtained in cooperation with L. Zajíček and can be found in entirety in our joint article [3].

We dealt with results concerning extendibility of a differentiable function f defined on a closed set $F \subset \mathbb{R}^n$ to a differentiable function on $\mathbb{R}^n$. Probably the most known result of this type is the special case of the famous Whitney's Extension Theorem which concerns extendibility to a C^1 function on $\mathbb{R}^n$ (see [2, Whitney's Extension Theorem] or [3, Theorem W]). In 1985, V. Aversa, M. Laczkovich and D. Preiss proved a result concerning extendibility to a differentiable (not necessarily C^1) function on $\mathbb{R}^n$ (see [1] or [3, Theorem ALP]). We proved another extension result (Theorem 1 below) that is a natural joint generalization of these two theorems. Roughly, it can be described as a theorem on extendibility to a differentiable function with preserving the continuity of the derivative. In its formulation, we use the notion of a (relative) strict derivative recalled in the following definition.

Definition (Strict derivative)**.** Let $\emptyset \neq F \subset \mathbb{R}^n$ be an arbitrary set and $f : F \to \mathbb{R}$ a function.

- We will say that $L_a \in \mathbb{R}^n$ is a *strict derivative of* f *at* $a \in F$ (with respect to F) if either $a \in \operatorname{der} F$ and

  $$\lim_{\substack{y\to a\\ x\to a\\ x,y\in F,\, x\neq y}} \frac{f(y) - f(x) - L_a \cdot (y - x)}{|y - x|} = 0 \quad \text{(with } x = a,\ y = a \text{ allowed)},$$

  or a is an isolated point of F.

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- We say that $L : F \to \mathbb{R}^n$ is a (relative) *strict derivative* of f if $L(a)$ is a strict derivative of f at a for each $a \in F$.

Our main extension result can be stated as follows:

Theorem 1. [3, Theorem 3.1] *Let $\emptyset \neq F \subset \mathbb{R}^n$ be a closed set, $f : F \to \mathbb{R}$ a function and $L : F \to \mathbb{R}^n$ a derivative of f such that $L \in B_1(F)$. Then there exists a function $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *$\overline{f}$ is differentiable on $\mathbb{R}^n$,*

(ii) *$\overline{f}(x) = f(x)$ and $(\overline{f})'(x) = L(x)$ for $x \in F$,*

(iii) *if $a \in F$, L is continuous at a and $L(a)$ is a strict derivative of f at a, then $(\overline{f})'$ is continuous at a,*

(iv) *$\overline{f}$ is C^∞ on $\mathbb{R}^n \setminus F$.*

We also concentrated on extending from special closed subsets of $\mathbb{R}^n$ that have in some sense large contingent cones in non-isolated points (see condition (C) introduced in Theorem 2). Recall the definition of tangent vectors and a contingent cone:

Definition (Tangent vectors and contingent cone)**.** Let $H \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is called *a tangent vector to H at x* if there exist $\{x_k\}_{k=1}^\infty \subset H$ and $\{\alpha_k\}_{k=1}^\infty \subset [0,\infty)$ such that $x_k \to x$ and $\alpha_k(x_k - x) \to v$. The set of all tangent vectors to H at x is called *a contingent cone of H at x* and will be denoted by $\mathrm{Tan}(H,x)$.

Note that whenever $H \subset \mathbb{R}^n$ and $a \in H$, then the derivative $f'(a)$ is determined uniquely for every function $f : H \to \mathbb{R}$ differentiable at a if and only if $\mathrm{Tan}(H,a)$ spans $\mathbb{R}^n$ (see [1, Corollary 2]).

We proved the following result:

Theorem 2. [3, Theorem 4.6] *Let $\emptyset \neq F \subset \mathbb{R}^n$ be a closed set, $f : F \to \mathbb{R}$ a function and let a derivative of f at x exist for every $x \in \operatorname{der} F$. Moreover, let the following condition hold:*

(C) *for every $x \in \operatorname{der} F$ there exist $r_x, d_x > 0$ such that*

$$\inf\{\sup\{|\det(v_1,\dots,v_n)| : v_1,\dots,v_n \text{ unit vectors from } \mathrm{Tan}(F,y)\} : y \in \operatorname{der} F \cap B(x,r_x)\} > d_x.$$

Then there exists a function $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ such that

(i) *$\overline{f}(x) = f(x)$ for every $x \in F$,*

(ii) $\overline{f}$ *is differentiable on* $\mathbb{R}^n$ *and* $\overline{f}$ *is* C^∞ *on* $\mathbb{R}^n \setminus F$*,*

(iii) if either $a \in \operatorname{der} F$ *and* f *is strictly differentiable at* a*, or* $a \in F \setminus \operatorname{der} F$*, then* $(\overline{f})'$ *is continuous at* a*.*

This theorem yields a simply formulated C^1 extension result. Since for $n = 1$ the condition (C) is automatically satisfied, this result is in such a case merely a reformulation of the original result obtained by H. Whitney in 1934 (compare with [4, Theorem I]):

Corollary. [3, Corollary 4.7] *Let* $\emptyset \neq F \subset \mathbb{R}^n$ *be a closed set,* $f : F \to \mathbb{R}$ *a function, let a strict derivative of* f *at* x *exist for every* $x \in \operatorname{der} F$ *and let the condition* (C) *hold. Then there exists a* C^1 *extension of* f *on* $\mathbb{R}^n$*.*

We further investigated the role of condition (C). If we replace this condition in the previous corollary by the assumption that $\operatorname{Tan}(F, x)$ spans $\mathbb{R}^n$ for every $x \in \operatorname{der} F$, it is still possible to prove the existence of a differentiable extension of f (see Proposition 3 below). However, even a violation of condition (C) at merely one point suffices us to construct a counterexample to the existence of a continuously differentiable extension of f (see [3, Example 4.14]).

Proposition 3. [3, Proposition 4.10] *Let* $\emptyset \neq F \subset \mathbb{R}^n$ *be a closed set,* $f : F \to \mathbb{R}$ *a strictly differentiable function and let* $\operatorname{Tan}(F, x)$ *span* $\mathbb{R}^n$ *for every* $x \in \operatorname{der} F$*. Then there exists a differentiable extension of* f *defined on* $\mathbb{R}^n$*.*

References

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