Real Analysis Exchange
Summer Symposium XXXVI, 2012, pp. 30–35

Erik Talvila, Department of Mathematics and Statistics, University of the Fraser Valley, Chilliwack, BC V2R 0N3, Canada.
email: `Erik.Talvila@ufv.ca`

FOURIER SERIES WITH THE CONTINUOUS PRIMITIVE INTEGRAL

Abstract

Fourier series are considered for the space of periodic distributions that are the distributional derivative of a continuous function. This space of distributions is denoted $\mathcal{A}_c(\mathbb{T})$ and is a Banach space under the Alexiewicz norm, $\|f\|_{\mathbb{T}} = \sup_{|I|\leq 2\pi} |\int_I f|$, the supremum being taken over intervals of length not exceeding 2π. It contains the periodic functions integrable in the sense of Lebesgue and Henstock–Kurzweil. Many of the properties of L^1 Fourier series continue to hold for this larger space, with the L^1 norm replaced by the Alexiewicz norm. The Riemann–Lebesgue lemma takes the form $\hat{f}(n) = o(n)$ as $|n| \to \infty$. The convolution is defined for $f \in \mathcal{A}_c(\mathbb{T})$ and g a periodic function of bounded variation. There is the estimate $\|f * g\|_\infty \leq \|f\|_{\mathbb{T}}\|g\|_{\mathcal{B}V}$. For $g \in L^1(\mathbb{T})$, $\|f * g\|_{\mathbb{T}} \leq \|f\|_{\mathbb{T}}\|g\|_1$. The convolution of f with a sequence of summability kernels converges to f in the Alexiewicz norm. Let D_n be the Dirichlet kernel and let $f \in L^1(\mathbb{T})$. Then $\|D_n * f - f\|_{\mathbb{T}} \to 0$ as $n \to \infty$.

1 Introduction and notation.

In this talk we consider Fourier series on the unit circle, $\mathbb{T}$ (“T” for thircle). Proofs can be found in [4], where many further results also appear. Slides from the talk are available at `http://www.math.ualberta.ca/~etalvila/research.html`.

Progress in Fourier analysis has paralleled progress in theories of integration. We describe below the *continuous primitive integral.* This is an integral

Mathematical Reviews subject classification: Primary: 26A39, 42A16, 46F10
Key words: Fourier series, convolution, distributional integral, continuous primitive integral, Henstock–Kurzweil integral, Schwartz distribution, generalised function

that includes the Lebesgue, Henstock–Kurzweil and wide Denjoy integrals. It has a simple definition in terms of distributions. The space of distributions integrable in this sense is a Banach space under the Alexiewicz norm. Many properties of Fourier series that hold for L^1 functions continue to hold in this larger space with the L^1 norm replaced by the Alexiewicz norm.

We use the following notation for distributions. The space of *test functions* is $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R}) = \{\phi\colon\mathbb{R}\to\mathbb{R} \mid \phi\in C^\infty(\mathbb{R})$ and $\text{supp}(\phi)$ is compact$\}$. The *support* of function ϕ is the closure of the set on which ϕ does not vanish and is denoted $\text{supp}(\phi)$. Under usual pointwise operations $\mathcal{D}(\mathbb{R})$ is a linear space over field $\mathbb{R}$. In $\mathcal{D}(\mathbb{R})$ we have a notion of convergence. If $\{\phi_n\}\subset\mathcal{D}(\mathbb{R})$ then $\phi_n\to 0$ as $n\to\infty$ if there is a compact set $K\subset\mathbb{R}$ such that for each n, $\text{supp}(\phi_n)\subset K$, and for each $m\geq 0$ we have $\phi_n^{(m)}\to 0$ uniformly on K as $n\to\infty$. The *distributions* are denoted $\mathcal{D}'(\mathbb{R})$ and are the continuous linear functionals on $\mathcal{D}(\mathbb{R})$. For $T\in\mathcal{D}'(\mathbb{R})$ and $\phi\in\mathcal{D}(\mathbb{R})$ we write $\langle T,\phi\rangle\in\mathbb{R}$. For $\phi,\psi\in\mathcal{D}(\mathbb{R})$ and $a,b\in\mathbb{R}$ we have $\langle T, a\phi+b\psi\rangle = a\langle T,\phi\rangle + b\langle T,\psi\rangle$. And, if $\phi_n\to 0$ in $\mathcal{D}(\mathbb{R})$ then $\langle T,\phi_n\rangle\to 0$ in $\mathbb{R}$. Linear operations are defined in $\mathcal{D}'(\mathbb{R})$ by $\langle aS+bT,\phi\rangle = a\langle S,\phi\rangle + b\langle T,\phi\rangle$ for $S,T\in\mathcal{D}'(\mathbb{R})$; $a,b\in\mathbb{R}$ and $\phi\in\mathcal{D}(\mathbb{R})$. If $f\in L^1_{loc}$ then $\langle T_f,\phi\rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$ defines a distribution $T_f\in\mathcal{D}'(\mathbb{R})$. The integral exists as a Lebesgue integral. All distributions have derivatives of all orders that are themselves distributions. For $T\in\mathcal{D}'(\mathbb{R})$ and $\phi\in\mathcal{D}(\mathbb{R})$ the distributional derivative of T is T' where $\langle T',\phi\rangle = -\langle T,\phi'\rangle$. If $p\colon\mathbb{R}\to\mathbb{R}$ is a function that is differentiable in the pointwise sense at $x\in\mathbb{R}$ then we write its derivative as $p'(x)$. For $x\in\mathbb{R}$ define the *translation* τ_x on distribution $T\in\mathcal{D}'(\mathbb{R})$ by $\langle\tau_x T,\phi\rangle = \langle T,\tau_{-x}\phi\rangle$ for test function $\phi\in\mathcal{D}(\mathbb{R})$ where $\tau_x\phi(y) = \phi(y-x)$. A distribution $T\in\mathcal{D}'(\mathbb{R})$ is *periodic* if $\langle\tau_p T,\phi\rangle = \langle T,\phi\rangle$ for some $p>0$ and all $\phi\in\mathcal{D}(\mathbb{R})$. The least such positive p is the *period*.

The Lebesgue integral of $f:\mathbb{R}\to\mathbb{R}$ is characterised by saying there is an absolutely continuous function F (the primitive) such that $F'(x) = f(x)$ for almost all x. The integral is then $\int_a^b f(x)\,dx = F(b)-F(a)$. There is a similar definition of the Henstock–Kurzweil integral using a larger class of primitives. See [1]. The continuous primitive integral is defined using a continuous primitive function. Define the primitives by $\mathcal{B}_c(\mathbb{T}) = \{F\colon\mathbb{R}\to\mathbb{R} | F\in C(\mathbb{R}), F(-\pi)=0, F(x) = F(y)+nF(\pi)$ if $y\in[-\pi,\pi), x = y+2n\pi$ for $n\in\mathbb{Z}\}$. It is easy to see that $\mathcal{B}_c(\mathbb{T})$ is a Banach space under the uniform norm $\|F\|_{\mathbb{T},\infty} = \sup_{|\alpha-\beta|\leq 2\pi}|F(\alpha)-F(\beta)|$. The integrable distributions are then given by $\mathcal{A}_c(\mathbb{T}) = \{f\in\mathcal{D}'(\mathbb{R}) \mid f = F'$ for some $F\in\mathcal{B}_c(\mathbb{T})\}$. For $a,b\in\mathbb{R}$ the integral of $f\in\mathcal{A}_c(\mathbb{T})$ is $\int_a^b f = F(b)-F(a)$ where $F\in\mathcal{B}_c(\mathbb{T})$ and $F' = f$. The distributional differential equation $T' = 0$ has only constant solutions and we have made our primitives in $\mathcal{B}_c(\mathbb{T})$ vanish at $-\pi$ so the primitive of a distribution in $\mathcal{A}_c(\mathbb{T})$ is unique. See [2] for more on the continuous primitive integral.

If $f:\mathbb{R}\to\mathbb{R}$ is a periodic function that is locally integrable in the Lebesgue, Henstock–Kurzweil or wide Denjoy sense then $T_f\in\mathcal{A}_c(\mathbb{T})$. This follows since the primitives for these integrals are continuous functions. As well, if $F\in\mathcal{B}_c(\mathbb{T})$ is a function of Weierstrass type that is continuous but has a pointwise derivative nowhere then the distributional derivative of F exists and $F'\in\mathcal{A}_c(\mathbb{T})$. If F is a continuous singular function, so that $F'(x)=0$ a.e., then $F'\in\mathcal{A}_c(\mathbb{T})$ and the continuous primitive integral is $\int_a^b F'=F(b)-F(a)$. In this case, $F'\in L^1(\mathbb{T})$ but the Lebesgue integral gives $\int_a^b F'(x)\,dx=0$.

The *Alexiewicz norm* of $f\in\mathcal{A}_c(\mathbb{T})$ is $\|f\|_{\mathbb{T}}=\sup_{|I|\leq 2\pi}|\int_I f|$, the supremum being taken over intervals of length not exceeding 2π. We have $\|f\|_{\mathbb{T}}=\|F\|_{\mathbb{T},\infty}=\max_{|\beta-\alpha|\leq 2\pi}|F(\beta)-F(\alpha)|$ where $F\in\mathcal{B}_c(\mathbb{T})$ is the primitive of f. The integral provides a linear isometry and isomorphism between $\mathcal{A}_c(\mathbb{T})$ and $\mathcal{B}_c(\mathbb{T})$. Define $\Phi:\mathcal{A}_c(\mathbb{T})\to\mathcal{B}_c(\mathbb{T})$ by $\Phi[f](x)=\int_{-\pi}^x f$. Then Φ is a linear bijection and $\|f\|_{\mathbb{T}}=\|\Phi[f]\|_{\mathbb{T},\infty}$. Hence, $\mathcal{A}_c(\mathbb{T})$ is a Banach space. The spaces of periodic Lebesgue, Henstock–Kurzweil and wide Denjoy integrable functions are all subspaces of $\mathcal{A}_c(\mathbb{T})$ but are not complete in the Alexiewicz norm. The space $\mathcal{A}_c(\mathbb{T})$ furnishes their completion.

The multipliers and dual space of $\mathcal{A}_c(\mathbb{T})$ are given by the periodic functions of bounded variation, $\mathcal{BV}(\mathbb{T})$. Using a Riemann–Stieltjes integral, the integration by parts formula is

$$\int_{-\pi}^x fg=F(x)g(x)-\int_{-\pi}^x F(t)\,dg(t),\quad x\in[-\pi,\pi). \tag{1}$$

From this we get the following version of the Hölder inequality.

Proposition 1 (Hölder inequality)**.** *Let $f\in\mathcal{A}_c(\mathbb{T})$. If $g\in\mathcal{BV}(\mathbb{T})$ then* $\left|\int_{-\pi}^{\pi} fg\right|\leq|\int_{-\pi}^{\pi} f|\inf|g|+\|f\|_{\mathbb{T}}Vg\leq\|f\|_{\mathbb{T}}(\|g\|_\infty+Vg)$.

2 Fourier coefficients.

Let $e_n(t)=e^{int}$. If $f\in\mathcal{A}_c(\mathbb{T})$ then the Fourier coefficients of f are $\hat{f}(n)=\int_{-\pi}^{\pi} fe_{-n}=\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$, where $n\in\mathbb{Z}$. Let $F(x)=\int_{-\pi}^x f$ be the primitive of f. Integrating by parts as in (1) gives

$$\hat{f}(n)=(-1)^nF(\pi)+in\int_{-\pi}^{\pi} F(t)e^{-int}\,dt. \tag{2}$$

This last integral is the Riemann integral of a continuous function. Formula (2) can be used as an alternative definition of $\hat{f}(n)$.

Theorem 2. *Let* $f \in \mathcal{A}_c(\mathbb{T})$*. Then (a)* $|\hat{f}(n)| \leq |F(\pi)| + |n| \int_{-\pi}^{\pi} |F|$ *where* $F(x) = \int_{-\pi}^{x} f$ *(b) for* $n \neq 0$*,* $|\hat{f}(n)| \leq 4\sqrt{2}\,|n| \|f\|_{\mathbb{T}}$ *(c)* $\hat{f}(n) = o(n)$ *as* $|n| \to \infty$ *and this estimate is sharp.*

Part (c) is a version of the Riemann–Lebesgue lemma for the continuous primitive integral.

The next theorem shows that when we have a sequence converging in the Alexiewicz norm, the Fourier coefficients also converge.

Theorem 3. *For* $j \in \mathbb{N}$*, let* $f, f_j \in \mathcal{A}_c(\mathbb{T})$ *such that* $\|f_j - f\|_{\mathbb{T}} \to 0$ *as* $j \to \infty$*. Then for each* $n \in \mathbb{Z}$ *we have* $\hat{f}_j(n) \to \hat{f}(n)$ *as* $j \to \infty$*. The convergence need not be uniform in* $n \in \mathbb{Z}$*.*

3 Convolution.

For $f \in \mathcal{A}_c(\mathbb{T})$ and $g \in \mathcal{BV}(\mathbb{T})$ the convolution is $\int_{-\pi}^{\pi} (f \circ r_x) g$ where $r_x(t) = x - t$. We write this as $f * g(x) = \int_{-\pi}^{\pi} f(x-t) g(t)\, dt$. This integral exists for all such f and g. The convolution inherits smoothness properties from f and g. The convolution was considered for the continuous primitive integral on the real line in [3]. Many of the results of that paper are easily adapted to the setting of $\mathbb{T}$, especially differentiation and integration theorems which we do not reproduce here.

Theorem 4. *Let* $f \in \mathcal{A}_c(\mathbb{T})$ *and let* $g \in \mathcal{BV}(\mathbb{T})$*. Then (a)* $f * g \in C(\mathbb{T})$ *(b)* $f * g = g * f$ *(c)* $\|f * g\|_\infty \leq \|f\|_{\mathbb{T}} \|g\|_{\mathcal{BV}}$ *(d) for* $y \in \mathbb{R}$ *we have* $\tau_y(f * g) = (\tau_y f) * g = f * (\tau_y g)$*. (e) If* $h \in L^1(\mathbb{T})$ *then* $f * (g * h) = (f * g) * h \in C(\mathbb{T})$*. (f) We have* $\widehat{f * g}(n) = \hat{f}(n) \hat{g}(n)$ *for all* $n \in \mathbb{Z}$*.*

Convolutions can also be defined for $f \in \mathcal{A}_c(\mathbb{T})$ and $g \in L^1(\mathbb{T})$ using the density of $L^1(\mathbb{T})$ in $\mathcal{A}_c(\mathbb{T})$. See [4] for details.

4 Convergence.

The series $\sum_{-\infty}^{\infty} \hat{f}(n) e^{int}$ is known as the Fourier series of f. If f is a smooth enough function then the Fourier series of f converges to f pointwise. For example, if the pointwise derivative $f'(x)$ exists then the Fourier series converges to f at x.

First we consider summability kernels.

Definition 5. *A summability kernel is a sequence* $\{k_n\} \subset \mathcal{BV}(\mathbb{T})$ *such that* $\int_{-\pi}^{\pi} k_n = 1$*,* $\lim_{n\to\infty} \int_{|s|>\delta} |k_n(s)|\, ds = 0$ *for each* $0 < \delta \leq \pi$ *and there is* $M \in \mathbb{R}$ *so that* $\|k_n\|_1 \leq M$ *for all* $n \in \mathbb{N}$*.*

Theorem 6. *Let $f \in \mathcal{A}_c(\mathbb{T})$. Let k_n be a summability kernel. Then $\|f * k_n - f\|_{\mathbb{T}} \to 0$ as $n \to \infty$.*

A commonly used summability kernel is the Fejér kernel,

$$k_n(t) = \frac{1}{2\pi} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt} = \frac{1}{2\pi(n+1)} \left[\frac{\sin((n+1)t/2)}{\sin(t/2)}\right]^2.$$

Lemma 7. *Let $f \in \mathcal{A}_c(\mathbb{T})$. Then $f * e_n(x) = \hat{f}(n)e^{inx}$. Let $g(t) = \sum_{-n}^{n} a_k e_k(t)$ for a sequence $\{a_k\} \subset \mathbb{R}$. Then $f * g(x) = \sum_{-n}^{n} a_k \hat{f}(k) e^{ikx}$.*

The proof follows from the identity $e_n(x-t) = e_n(x)e_n(-t)$ and linearity of the integral.

The lemma allows us to prove that trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$ and gives a uniqueness result. Let k_n be the Fejér kernel and define $\sigma_n[f] = k_n * f$. From Theorem 6 we have $\sigma_n[f] \to f$ in the Alexiewicz norm. The Lemma shows $\sigma_n[f]$ is a trigonometric polynomial. Hence, the trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$.

Theorem 8. *Let $f \in \mathcal{A}_c(\mathbb{T})$. The trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$;*

$$\sigma_n[f](t) = \frac{1}{2\pi} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k) e^{ikt} \text{ and } \lim_{n\to\infty} \|f - \sigma_n[f]\|_{\mathbb{T}} = 0. \quad (3)$$

If $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$ then $f = 0$.

For $n \geq 0$ define the Dirichlet kernel $D_n(t) = \sum_{-n}^{n} e^{ikt} = \sin[(n+1/2)t]/\sin(t/2)$. Notice that according to the definition in Theorem 6, D_n is not a summability kernel. In fact, $\|D_n\|_1 \sim (4/\pi^2)\log(n)$ as $n \to \infty$. However, $\|D_n\|_{\mathbb{T}}$ are bounded. This shows that $D_n * f$ converges to f in $\|\cdot\|_{\mathbb{T}}$ for $f \in L^1(\mathbb{T})$.

Theorem 9. *The sequence $\|D_n\|_{\mathbb{T}}$ is bounded. Let $f \in L^1(\mathbb{T})$. Then $\|D_n * f - f\|_{\mathbb{T}} \to 0$ as $n \to \infty$.*

Since the Dirichlet kernels are not uniformly bounded in the L^1 norm there is a function $f \in \mathcal{A}_c(\mathbb{T})$ such that $\|D_n * f - f\|_{\mathbb{T}} \not\to 0$. See [4] for an example. This example, together with Theorem 9 shows the value of the Alexiewicz norm, even for L^1 functions.

References

[1] R.A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, American Mathematical Society, Providence, 1994.

[2] E. Talvila, *The distributional Denjoy integral*, Real Anal. Exchange **33** (2008), 51–82.

[3] E. Talvila, *Convolutions with the continuous primitive integral*, Abstr. Appl. Anal. **2009** (2009) Art. ID 307404, 18 pp.

[4] E. Talvila, *Fourier series with the continuous primitive integral*, Journal of Fourier analysis and applications, **18** (2012), 27–44.