Real Analysis Exchange
Summer Symposium 2010, pp. 35–38

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ARE FUNCTIONS WITH A MONOTONE GRAPH SMOOTH?

Abstract

A metric space (X, d) is called *monotone* if there is a linear order $<$ on X and $c > 0$ such that $d(x, y) \leqslant c\, d(x, z)$ for all $x < y < z$ in X. We present a brief review of properties of graphs of functions with a monotone graph.

1 Introduction.

Monotone metric spaces were introduced in [4, 3]. By one of the equivalent definitions, given $c \geqslant 1$, a metric space (X, d) is called *c-monotone* if there is a linear order $<$ on X such that if $x < y < z$, then

$$\max\bigl(d(x, y), d(y, z)\bigr) \leqslant c\, d(x, z).$$

The order $<$ and constant c are termed the *witnessing order* and *witnessing constant*, respectively. A metric space is called monotone if it is c-monotone for some c, and σ-monotone if it is a countable union of monotone spaces.

Quite a number of papers are written or under preparation on the subject: [4, 5, 6, 2, 3, 1]. A brief review is given in my last year report.

In this report we focus on properties of continuous functions that have a monotone or σ-monotone graph.

Let us mention some properties of monotone sets in Euclidean spaces. $\mathcal{H}^m$ and $\dim_{\mathsf{H}}$ denote, respectively, the m-dimensional Hausdorff measure and Hausdorff dimension.

Mathematical Reviews subject classification: Primary: 26A24, 26A27; Secondary: 28A80
Key words: monotone space, continuous function, nowhere differentiable function, absolutely continuous function
*Work on this project was conducted during the author's sabbatical stay at the Instituto de matemáticas, Unidad Morelia, Universidad Nacional Autonóma de México supported by CONACyT grant no. 125108.

Theorem 1.1. *If $X \subseteq \mathbb{R}^n$ is c-monotone, then* $\dim_{\mathsf{H}} X \leqslant n - \dfrac{q}{c\ln(c+1)}$, *where q is an absolute constant. In particular,* $\dim_{\mathsf{H}} X < n$.

Corollary 1.2. *If $X \subseteq \mathbb{R}^n$ is σ-monotone, then it is Lebesgue-null.*

Recall that a set $A \subseteq \mathbb{R}^n$ is *m-rectifiable* if $\mathcal{H}^m$-almost all points of A can be covered by countably many C^m-surfaces, and that $B \subseteq \mathbb{R}^n$ is *purely m-unrectifiable* if $\mathcal{H}^m(X \cap A) = 0$ for each m-rectifiable set $A \subseteq \mathbb{R}^n$.

Corollary 1.3. *A σ-monotone set $X \subseteq \mathbb{R}^n$ is purely m-unrectifiable for each $m \geqslant 2$.*

Recall that a set $X \subseteq \mathbb{R}^n$ is *lower porous* if there is $p > 0$ such that for every $x \in \mathbb{R}^n$ and every $r > 0$ there is $y \in \mathbb{R}^n$ such that $B(y, pr) \subseteq B(x, r) \setminus X$.

Theorem 1.4. *Every monotone set in $\mathbb{R}^n$ is lower porous.*

On the other hand, monotone sets approximate any Borel set in the following sense.

Theorem 1.5. *Every Borel set $B \subseteq \mathbb{R}^n$ contains a σ-monotone set $X \subseteq B$ such that* $\dim_{\mathsf{H}} X = \dim_{\mathsf{H}} B$.

2 Functions with a monotone graph are not nowhere differentiable.

A *curve* in $\mathbb{R}^n$ is an image of a one-to-one continuous mapping $\psi : [0,1] \to \mathbb{R}^n$. The mapping ψ is a *parametrization* of C. Note that since any curve C is connected, there are only two linear orders that can witness monotonicity of C. Moreover one of them is the reverse of the other. Since being c-monotone is invariant with respect to reversing the witnessing order, it does not matter which of them we choose. Overall, given a curve C and (any) parametrization ψ of of C, the curve C is c-monotone if and only if for all $x < y < z \in [0,1]$

$$|\psi(x) - \psi(y)| \leqslant c|\psi(x) - \psi(z)|, \tag{1}$$

$$|\psi(z) - \psi(y)| \leqslant c|\psi(x) - \psi(z)|. \tag{2}$$

Hence restricting ourselves to curves, the formula defining monotonicity is remarkably less complex.

It is easy to show that a C^1-curve is σ-monotone. One may ask if there is a converse, i.e. if monotone curves possess some smoothness properties. It turns out that it is not so: The well-known von Koch curve is monotone and yet its Hausdorff dimension is strictly bigger than 1 and it does not have a tangent line at any point. The calculation of the precise value of the witnessing constant for von Koch curve is in progress.

But what if we further restrict ourselves to graphs of continuous functions? Such a graph is of course a curve. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Formally there is no difference between f and its graph, but confusion may arise for instance from "f is monotone". Therefore we use $\mathfrak{f}$ when referring to the graph of f as a pointset in the plane. Given $E \subseteq [0,1]$, denote $\mathfrak{f}|E$ the graph of f restricted to E.

Theorem 2.1. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function with a monotone graph. Then $\mathscr{H}^1(\mathfrak{f})$ is σ-finite. In particular,* $\dim_{\mathsf{H}} \mathfrak{f} = 1$.

Theorem 2.2. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function with a monotone graph. Then there are Borel sets A, B such that $A \cup B = [0,1]$ and*

(i) $\mathscr{H}^1(\mathfrak{f}|A) < \infty$,

(ii) *$f'(x)$ exists for each $x \in B$.*

Corollary 2.3. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function with a σ-monotone graph. Then f is differentiable at a set that meets every interval at a perfect set.*

So monotone graphs of continuous functions indeed possess some weak smoothness properties that curves may lack. Can one hope that the latter corollary can be strengthened to "continuous function with a monotone graph is differentiable almost everywhere"?

3 Non-differentiable functions with a monotone graph.

The answer is "no": Immediately after my presentation of the above results at the conference a discussion started. Within two days Tamás Mátrai came up with an idea that was developed by Václav Vlasák. Their efforts lead to an example of a continuous function that has a monotone graph and yet it is almost nowhere differentiable.

Theorem 3.1. *For any $c > 1$ there is a continuous function $f : [0,1] \to \mathbb{R}$ such that*

(i) *the graph of f is c-monotone,*

(ii) *both one-sided approximate derivatives fail to exist almost everywhere.*

In particular, f is almost nowhere differentiable.

So Theorem 2.2 and Corollary 2.3 is the best one can get for differentiability of functions with a monotone graph.

One may ask if there is such an example even for $c = 1$. Discussion with Aleš Nekvinda resulted in a negative answer:

Proposition 3.2. *If f is continuous and $\mathfrak{f}$ is* 1*-monotone, then f is differentiable almost everywhere.*

4 Absolutely continuous function with a non-σ-monotone graph.

So monotone graph does not imply differentiability. When I got back to Mexico, I discussed with Michael Hrušák the dual question: Does differentiability ensure a σ-monotone graph?

Theorem 4.1. *There is an absolutely continuous function whose graph is not σ-monotone.*

The details, proofs and some more results will appear in a paper that is in the final stage of preparation. The approximate set of authors is: me, Pieter Allaart, Michael Hrušák, Tamás Mátrai, Aleš Nekvinda and Václav Vlasák.

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