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RESULTS FOR SLOWLY OSCILLATING CONTINUOUS FUNCTIONS

As per usual, let $\mathbb{R}$ denote the set of real numbers. Although many of these definitions use subsets of the real line or functions whose domain is a subset of the real line, we will state all the definitions for the real line. These first definitions come from Çakalli ([3]).

Definition 1. *A sequence* $\{x_n\}$ *of points in* $\mathbb{R}$ *is called* slowly oscillating *if*

$$\lim_{\lambda\to 1^+}\left[\overline{\lim_{n\to\infty}}\left(\max_{n+1\le k\le[\lambda n]}|x_k-x_n|\right)\right]=0 \tag{1}$$

where $[\lambda n]$ *refers to the integer part of* λn.

There follows a more usual definition using the ε's, δ's,and N's typical for generalizations of the properties of a sequence.

Definition 2. *A sequence* $\{x_n\}$ *of points in* $\mathbb{R}$ *is slowly oscillating if for any* $\varepsilon > 0$ *there exists a positive real number* $\delta = \delta(\varepsilon)$ *and a natural number* $N = N(\varepsilon)$ *such that*

$$|x_m - x_n| < \varepsilon \tag{2}$$

if $n \geq N$ *and* $n \leq m \leq (1+\delta)\, n$.

Our first result is to show that slowly oscillating is in fact different and worth studying.

Example 1. *There is a sequence in* $\mathbb{R}$ *which is slowly oscillating, but not Cauchy.*

The goal in Çakalli's article was to introduce the concept of a slowly oscillating continuous function.

Definition 3. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is* slowly oscillating continuous *if it transforms slowly oscillating sequences into slowly oscillating sequences.*

In [3], the author asserts that every slowly oscillating continuous function is, in fact, continuous in the ordinary sense. His idea is correct (though his proof flawed) and we next present a proof of his assertion.

Theorem 1. *If $f : \mathbb{R} \to \mathbb{R}$ is slowly oscillating continuous, then f is continuous in the ordinary sense.*

In the talk abstract [4], the author wrote "the purpose of this talk is to conjecture if the slowly oscillating continuity is equivalent to uniform continuity." He then proved that a uniformly continuous function must be slowly oscillating continuous. We present the other direction.

Theorem 2. *Let f be a continuous function defined on the real line. If f is not uniformly continuous, then f cannot be slowly oscillating continuous.*

Next we look at metric preserving functions ([7] or [11]).

Definition 4. *Let $\mathbb{R}^+$ denote the non-negative real numbers. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called* metric preserving *if for every metric space (X, ρ) the composite function $f \circ \rho$ is still a metric on X.*

The most well-known example of this, one found as a homework exercise in most books which introduce metric spaces, is

$$f(x) = \frac{x}{x+1} \tag{3}$$

which turns any metric into a bounded metric.

We connect metric preserving functions and slowly oscillating continuous functions as follows:

Theorem 3. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a metric preserving function which is continuous at the origin and increasing in some nondegenerate interval $[0, \beta]$. Then the function $f^* : \mathbb{R} \to \mathbb{R}$ given by*

$$f^*(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases} \tag{4}$$

is a slowly oscillating continuous function.

Therefore we have a new collection of slowly oscillating continuous functions: those based on metric preserving functions. Since there are already examples of metric preserving functions which meet our criteria and have unusual properties, we get some pathological examples of slowly oscillating continuous functions.

Example 2. *[6] There exists a slowly oscillating continuous function which is nowhere differentiable.*

Example 3. *[8] There exists a slowly oscillating continuous function which is increasing yet $f'(x) = 0$ almost everywhere.*

Example 4. *[10] For any measure zero, $\mathcal{G}_\delta$ set Z there exists a slowly oscillating continuous function which is everywhere differentiable (in the extended sense) such that $\{x : |f'(x)| = \infty\} = Z \cup \{0\}$.*

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