Real Analysis Exchange
Summer Symposium 2010, pp. 44–47

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BANACH ALGEBRA NORMS FOR SPACES OF FUNCTIONS OF GENERALIZED BOUNDED VARIATION

Abstract

The purpose of this note is to offer an elementary proof of the submultiplicativity of canonical norms on certain space of functions of generalized bounded variation.

This is a summary of a talk delivered at the Summer Symposium in Real Analysis – The Buckeye Symposium – at the College of Wooster in July 2010. The paper [3] upon which it is based will appear in the Real Analysis Exchange. For convenience, the bibliography at the end of this note is the full bibliography from that paper.

Throughout this overview, I denotes a compact interval of the real line, p is a real number that satisfies $1 \leq p$, and $\Lambda = (\lambda_j)_{j=1}^{\infty}$ is a non-decreasing sequence of positive real numbers for which the series $\sum_{j=1}^{\infty} 1/\lambda_j$ diverges. For any natural number n, consider subintervals $I_j = [a_j, b_j]$, $j = 1, 2, ..., n$, of I that are non-overlapping in the sense that their interiors are pairwise disjoint. When f is a real-valued function defined on I, form the sum $\sum_{j=1}^{n} |f(I_j)|^p/\lambda_j$, where the notation $f(I_j)$ represents the difference $f(b_j) - f(a_j)$. The Λ_p-variation of f, denoted $V_{\Lambda_p}(f)$, is defined to be the supremum over all sums of this type, and, if $V_{\Lambda_p}(f) < \infty$, then f is said to be a function of bounded Λ_p-variation. The collection of all functions of bounded Λ_p-variation on I constitutes the linear space $\Lambda_p BV$. Whenever $f \in \Lambda_p BV$, it follows that f is bounded (Lemma 1.6 of [12]), and the definition

$$\|f\|_{\Lambda_p} = \|f\|_\infty + V_{\Lambda_p}(f)^{1/p}$$

provides a complete norm for $\Lambda_p BV$. We show that this norm is submultiplicative.

Mathematical Reviews subject classification: Primary: 26A45; Secondary: 46J10
Key words: bounded variation, Banach algebra

Theorem 1. *If $f, g \in \Lambda_p BV$, then $fg \in \Lambda_p BV$ and $\|fg\|_{\Lambda_p} \leq \|f\|_{\Lambda_p}\|g\|_{\Lambda_p}$.*

PROOF. Let $f, g \in \Lambda_p BV$ be given. For any natural number n and any collection of non-overlapping subintervals $I_1, I_2, \ldots, I_n$ of I, we have

$$\begin{aligned}
\sum_{j=1}^{n} \frac{|fg(I_j)|^p}{\lambda_j} &= \sum_{j=1}^{n} \frac{|fg(b_j) - fg(a_j)|^p}{\lambda_j} \\
&= \sum_{j=1}^{n} \frac{\big|f(b_j)g(b_j) - f(a_j)g(b_j) + f(a_j)g(b_j) - f(a_j)g(a_j)\big|^p}{\lambda_j} \\
&= \sum_{j=1}^{n} \frac{\big|g(b_j)f(I_j) + f(a_j)g(I_j)\big|^p}{\lambda_j} \\
&\leq \sum_{j=1}^{n} \frac{(\|g\|_\infty |f(I_j)| + \|f\|_\infty |g(I_j)|)^p}{\lambda_j} \\
&= \sum_{j=1}^{n} \left(\frac{\|g\|_\infty |f(I_j)|}{\lambda_j^{1/p}} + \frac{\|f\|_\infty |g(I_j)|}{\lambda_j^{1/p}} \right)^p \\
&\leq \left[\left(\sum_{j=1}^{n} \frac{\|g\|_\infty^p |f(I_j)|^p}{\lambda_j} \right)^{1/p} + \left(\sum_{j=1}^{n} \frac{\|f\|_\infty^p |g(I_j)|^p}{\lambda_j} \right)^{1/p} \right]^p \\
&= \left[\|g\|_\infty \left(\sum_{j=1}^{n} \frac{|f(I_j)|^p}{\lambda_j} \right)^{1/p} + \|f\|_\infty \left(\sum_{j=1}^{n} \frac{|g(I_j)|^p}{\lambda_j} \right)^{1/p} \right]^p \\
&\leq \left[\|g\|_\infty V_{\Lambda_p}(f)^{1/p} + \|f\|_\infty V_{\Lambda_p}(g)^{1/p} \right]^p.
\end{aligned}$$

The third-to-last step obtains from Minkowski's inequality applied to the n-tuples

$$\left(\frac{\|g\|_\infty |f(I_1)|}{\lambda_1^{1/p}}, \ldots, \frac{\|g\|_\infty |f(I_n)|}{\lambda_n^{1/p}} \right) \quad \text{and} \quad \left(\frac{\|f\|_\infty |g(I_1)|}{\lambda_1^{1/p}}, \ldots, \frac{\|f\|_\infty |g(I_n)|}{\lambda_n^{1/p}} \right).$$

It follows that $fg \in \Lambda_p BV$ and $V_{\Lambda_p}(fg)^{1/p} \leq \|g\|_\infty V_{\Lambda_p}(f)^{1/p} + \|f\|_\infty V_{\Lambda_p}(g)^{1/p}$.

From this inequality, prominently featured in section 4 of [6], it follows that

$$\begin{aligned}\|fg\|_{\Lambda_p} &= \|fg\|_\infty + V_{\Lambda_p}(fg)^{1/p} \\ &\leq \|f\|_\infty \|g\|_\infty + \|g\|_\infty V_{\Lambda_p}(f)^{1/p} + \|f\|_\infty V_{\Lambda_p}(g)^{1/p} \\ &\leq \|f\|_\infty \|g\|_\infty + \|g\|_\infty V_{\Lambda_p}(f)^{1/p} + \|f\|_\infty V_{\Lambda_p}(g)^{1/p} + V_{\Lambda_p}(f)^{1/p} V_{\Lambda_p}(g)^{1/p} \\ &= \left(\|f\|_\infty + V_{\Lambda_p}(f)^{1/p}\right)\left(\|g\|_\infty + V_{\Lambda_p}(g)^{1/p}\right) \\ &= \|f\|_{\Lambda_p} \|g\|_{\Lambda_p},\end{aligned}$$

to establish the result. □

A cousin of the function space $\Lambda_p BV$ is the space consisting of sequences $x = (x_j)_{j=0}^\infty$ for which the sum $\sum_{j=1}^\infty |x_j - x_{j-1}|^p/\lambda_j$ is finite. It is tempting to introduce a norm by defining

$$\|x\| = \|x\|_\infty + \left(\sum_{j=1}^\infty \frac{|x_j - x_{j-1}|^p}{\lambda_j}\right)^{1/p}.$$

The simplest case, when $p = 1$ and Λ is the constant sequence of all 1's, results in the well-known Banach algebra *bv* of sequences of bounded variation. On the other hand, when $p = 1$ and $\Lambda = (j)_{j=1}^\infty$, note that the resulting linear space *hbv* of sequences of bounded harmonic variation contains, for example, the unbounded sequence $x = (0, 1, \sqrt{2}, \sqrt{3}, ...)$. Thus, because it involves the supremum, the above candidate for a norm is not available for *hbv*. Similarly, for $p > 1$ and constant Λ sequence, there are unbounded sequences with bounded p-variation. Moreover, the square $(0, 1, 2, 3, ...)$ of the sequence x above fails to have finite harmonic variation. Thus, the issue of submultiplicativity of a norm becomes irrelevant for the sequence space *hbv* since it is not even stable under pointwise multiplication.

These problems are readily side-stepped by restricting attention to the *bounded sequences* of bounded p-variation. For these spaces, equipped with pointwise operations, submultiplicativity of the norms is obtained *mutatis mutandis.*

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