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VARIATIONS ON AXER'S THEOREM

1 Axer's Theorem

We will state Axer's theorem in the form given given by Hardy [2, p.378], which is a substantial generalization of the original result [1]. Although the theorem appears to be elementary, it may be used to deduce the prime number theorem from Wiener's Tauberian theorem and provides a tool to show the relationship between various important number-theoretic results. Many related results are to be found in an important paper of Ingham [3].

Theorem 1. *If*

(a) $f(x)$ *is of bounded variation in every finite interval* $[1, X]$,

(b) $A_x = \sum^x a_n = o(x)$,

and either of the pairs of conditions

(c1) $f(x) = O(1)$, (d1) $\sum^x |a_n| = O(x)$,

(c2) $f(x) = O(x^\alpha), 0 < \alpha < 1$, (d2) $a_n = O(1)$

is satisfied, then

$$\sum^x a_n f\left(\frac{x}{n}\right) = o(x).$$

Here $\sum\limits^x$ denotes summation over the integers from 1 to $[x]$, the greatest integer not exceeding x.

We consider the results that can be obtained by assuming that the function f is of generalized bounded variation, in particular, Λ-bounded variation and Φ-bounded variation. Although Hardy states that Landau has proven the converse result, the converse is actually false without an additional hypothesis on f. We give a counterexample and prove the amended converse theorem.

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2 Results

If we replace bounded variation by Λ-bounded variation we may generalize Axer's theorem in two different ways. We may strengthen the condition on A_x to obtain the original conclusion or use the original condition and obtain a weaker conclusion

Let $\Lambda = \{\lambda_n\}$. If in place of (b) we assume $A_n = o\left(n\lambda_n^{-1}\right)$,with the (c) and (d) conditions as before, then we obtain

$$\sum^{x} a_n f\left(\frac{x}{n}\right) = o(x).$$

If we take (b) as in Axer's theorem, we obtain

$$\sum^{x} a_n f\left(\frac{x}{n}\right) = o(x\lambda_{[x]}).$$

Now let us suppose that $\Phi(x)$ and $\Psi(x)$ are conjugate Young's functions, i.e., $\Phi(0) = 0$, Φ is continuous and non-decreasing for $x > 0$, and

$$\Psi(y) = \max_{x \geqslant 0} \{xy - \Phi(x)\}.$$

It is usual to assume stronger conditions on Φ, but that is not necessary here.

If we assume that $f(x)$ is of Φ-bounded variation in every finite interval $1 \leqslant x \leqslant X$, and

$$\sum^{x} \Psi\left(|A_n|\right) = o(x),$$

then with (c1) and (d1) as before, we have

$$\sum^{x} a_n f\left(\frac{x}{n}\right) = o(x).$$

Our converse result is the following.

Theorem 2. *If $f(x)$ is a function of bounded variation in every finite interval $[1, X]$, such that*

$$f(x) = O(1),$$
$$|f(x)| \geqslant c > 0,$$

and

$$\sum^{x} a_n f\left(\frac{x}{n}\right) = o(x), \text{ where } \sum^{x} |a_n| = O(x),$$

then

$$\sum^{x} a_n = o(x).$$

References

[1] A. Axer, *Beitrag zur Kenntnis der zahlentheoretischen Funktionen* $\mu(n)$ *und* $\lambda(n)$, Prace mat.-fiz., **21** (1910), 65–95.

[2] G. H. Hardy, *Divergent series,* Oxford at the Clarendon Press, 1948.

[3] A. E. Ingham, *Some Tauberian theorems connected with the prime number theorem,* J. London Math. Soc., **20** (1945), 171–180.