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A FEYNMAN-KAC SOLUTION TO A RANDOM IMPULSIVE EQUATION OF SCHRÖDINGER TYPE

Abstract

If a force is applied to a particle undergoing Brownian motion, the resulting motion has a state function which satisfies a diffusion or Schrödinger-type equation. We consider a process in which Brownian motion is replaced by a process which has Brownian transitions at all times other than random times at which the transitions have an additional "impulsive" displacement. Using a Feynman-Kac formulation based on generalized Riemann integration, we examine the resulting equation of motion.

1 Introduction.

This presentation is based on results contained in a joint work [1] with Marcia Federson and Patrick Muldowney.

When some system parameter has a discontinuity, the term "impulse" or "jump" can be a vivid way of describing this characteristic of the system.

Sometimes the state of a system can be described by a differential equation. For instance, a diffusion can be described by a parabolic partial differential equation satisfied by some function of displacement and time.

The purpose of this paper is to examine the relationship between discontinuities in the state function which characterizes the diffusion, and impulsive

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changes in the underlying diffusion itself. We use a Feynman-Kac formulation to show the connection between these two classes of discontinuities.

The scenario we tackle in this paper requires us to consider displacements x_t at various times t in some time interval $]\tau', \tau[$, and also to consider the possibility that, at arbitrary times $\tau' < t_1 < \cdots < t_{n-1} < \tau$, the displacements x_{t_j} satisfy $u_j \leq x_{t_j} \leq v_j$ for $1 \leq j \leq n-1$; or $x_j \in \mathrm{Cl}(I_j)$ (closure of I_j), where we write $I_j = [u_j, v_j[$ and $x_j = x_{t_j}$ for each j.

Writing $x = (x_t)_{t \in]\tau', \tau[}$ and $I = \{x : x_j \in I_j,\ 1 \leq j \leq n-1\}$, we are led to consider Riemann sums such as $\sum f(x)\mu(I)$. The corresponding integrals are $\int f(x)\mu(I)$. The domain of integration is the set $\{x\}$, where each x is a mapping of the form

$$x :]\tau', \tau[\mapsto \mathbb{R}, \quad \text{with} \quad x_t = x(t) \in \mathbb{R} \quad \text{for} \quad \tau' < t < \tau.$$

We denote this domain by $\mathbb{R}^{]\tau', \tau[}$, which can be viewed as a Cartesian product of $\mathbb{R}$ by itself uncountably many times. The partitioning intervals I are cylindrical subsets of $\mathbb{R}^{]\tau', \tau[}$.

The framework of generalized Riemann integration outlined above can be adapted to this scenario, and this is explained in more detail in [2].

Treating the elements x as sample paths in some version of the Brownian motion, we develop a Feynman-Kac representation

$$u(\xi, \tau) = \int_{\mathbb{R}^{]\tau', \tau[}} f(x)\mu(I),$$

with $\xi := x(\tau)$, of the solutions $u(\xi, \tau)$ of a partial differential equation

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} + V(\xi)u = 0,$$

where V is a potential function.

With the aid of this theoretical framework, we can relate discontinuities in $u(\xi, \tau)$ to "impulses" in the sample paths x.

2 Main Result.

Suppose $0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_p < \tau$ are given numbers and $\tau \in]0, +\infty[$. Define $\Delta = \mathbb{R} \times [0, \tau]$,

$$\Gamma_k = \{(\psi, t) : \psi \in \mathbb{R}, t \in]\tau_k, \tau_{k+1}[\}, \quad 0 \leq k \leq p-1,$$

$$\overline{\Gamma}_k = \{(\psi, t) : \psi \in \mathbb{R}, t \in [\tau_k, \tau_{k+1}[\}, \quad 0 \leq k \leq p-1,$$

$$\Gamma_p = \{(\psi,\, t):\, \psi \in \mathbb{R},\, t \in]\tau_p,\, \tau[\}\,, \quad \overline{\Gamma}_p = \{(\psi,\, t):\, \psi \in \mathbb{R},\, t \in [\tau_p,\, \tau[\}\,,$$

$$\Gamma = \bigcup_{k=0}^{p} \Gamma_k \quad \text{and} \quad \overline{\Gamma} = \bigcup_{k=0}^{p} \overline{\Gamma}_k.$$

Let $\mathcal{K}(\Delta,\, \mathbb{R})$ be the class of all functions $u:\, \Delta \to \mathbb{R}$ such that

i) the functions $u|_{\Gamma_k}$, $k = 0, 1, ..., p$, are continuous.

ii) for each k, $k = 1, ..., p$, the limit $\lim\limits_{(\nu,\, t)\to(\psi,\, \tau_k^-)} u(\nu,\, t) = u(\psi,\, \tau_k^-)$, $\psi \in \mathbb{R}$, exists.

iii) for each k, $k = 1, ..., p$, the limit $\lim\limits_{(\nu,\, t)\to(\psi,\, \tau_k^+)} u(\nu,\, t) = u(\psi,\, \tau_k^+)$, $\psi \in \mathbb{R}$, exists.

iv) for each k, $k = 1, ..., p$, we have $u(\psi,\, \tau_k) = u(\psi,\, \tau_k^+)$, $\psi \in \mathbb{R}$.

We consider the equation of Schrödinger type in Γ

$$\frac{\partial}{\partial t} u(\psi,\, t) - \frac{1}{2}\frac{\partial^2}{\partial \psi^2} u(\psi,\, t) + V(\psi) u(\psi,\, t) = 0, \tag{1}$$

subject to the impulse condition

$$u(\psi,\, \tau_k) - u(\psi,\, \tau_k^-) = \mathbb{I}(\psi,\, \tau_k,\, u(\psi,\, \tau_k)), \tag{2}$$

where $k = 1, 2, ..., p$, and $V : \mathbb{R} \to \mathbb{R}$ and $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}$ are functions taking real values and $\mathbb{I}$ is not identically zero.

Definition 2.1. *The function $u : \Delta \to \mathbb{R}$ is called a solution of the problem* $(1) - (2)$ *if:*

i) $u \in \mathcal{K}(\Delta,\, \mathbb{R})$*;*

ii) the derivatives $u_t(\psi,\, t)$ *and* $u_{\psi\psi}(\psi,\, t)$ *exist, for* $(\psi,\, t) \in \Gamma$*;*

iii) u *satisfies* (1) *in* Γ *and* (2) *at each* τ_k*,* $k = 1, 2, ..., p$*.*

Given $0 < \tau' < \tau$, we assume $\{\tau_p\}_{p\geq 1} \cap]\tau',\, \tau[\neq \emptyset$, where $\tau_j = \tau' + \sum_{i=1}^{j} \omega_i$, $j = 1, 2, \ldots$, and $\{\omega_i : i = 1, 2, \ldots\}$ is a sequence of random variables with $\omega_i \in]0,\, T[$, $0 < T \leq +\infty$, where ω_i is independent of ω_j when $i \neq j$ for all $i, j = 1, 2, \ldots$.

Let $U_{\mathcal{I}} : \mathbb{R} \to \mathbb{R}$ be a continuous function. Given $s \in]\tau', \tau[$ and $\varsigma \in \mathbb{R}$, let $N^{(s)}$ be the set $\{t_1, ..., t_{r-1}\}$, where $t_0 = \tau'$ and $t_r = s$ $(r = r(s) \in \mathbb{N})$. Then define $g_{\mathcal{I}}(x, N^{(s)}) = \prod_{j\in\mathcal{N}\setminus\mathcal{J}} \frac{\exp\left(-\frac{1}{2}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}\right)}{\sqrt{2\pi(t_j-t_{j-1})}} \prod_{j\in\mathcal{J}} \frac{\exp\left(-\frac{1}{2}\frac{(J_j(x_j)+x_j-x_{j-1})^2}{t_j-t_{j-1}}\right)}{\sqrt{2\pi(t_j-t_{j-1})}}$ and $q_{\mathcal{I}}(x, N^{(s)}, I^{(s)}) = g_{\mathcal{I}}(x, N^{(s)}) \prod_{j=1}^{r(s)-1} \Delta I_j$, where $I^{(s)} = I[N^{(s)}]$. Let $W_{\mathcal{I}}(x, N^{(s)}, I^{(s)}; \varsigma, s) = q_{\mathcal{I}}(x, N^{(s)}, I^{(s)}) e^{-U_{\mathcal{I}}(x_{r(s)-1})(s-\tau')}$.

Theorem 2.1. *Let $\tau' < s < \tau$ and $\varsigma \in \mathbb{R}$. The function*

$$\phi_{\mathcal{I}}(\varsigma, s) = \int_{\mathbb{R}^{]\tau', s[}} W_{\mathcal{I}}(x, N^{(s)}, I^{(s)}; \varsigma, s)$$

satisfies the partial differential equation of Schrödinger type in Γ

$$\frac{\partial}{\partial s}u(\varsigma, s) - \frac{1}{2}\frac{\partial^2}{\partial \varsigma^2}u(\varsigma, s) + U_{\mathcal{I}}(\varsigma)u(\varsigma, s) = 0,$$

subject to the impulse condition

$$u(\xi_k, \tau_k) - u(\xi_k, \tau_k^-) = \mathbb{I}(\xi_k, \tau_k, u(\xi_k, \tau_k)),$$

where $\tau_j = \tau' + \sum_{i=1}^{j} \omega_i$, $j = 1, 2, \ldots$, $\{\omega_i : i = 1, 2, \ldots\}$ *is a sequence of random variables with* $\omega_i \in]0, T[$, $0 < T \le +\infty$, ω_i *is independent of* ω_j *when* $i \neq j$ *for all* $i, j = 1, 2, \ldots$, $x(\tau_k) = \xi_k \in \mathbb{R}$ *and* $\mathbb{I}(\xi_k, \tau_k, u(\xi_k, \tau_k)) = \phi_{\mathcal{I}}(\xi_k, \tau_k) - \phi_{\mathcal{I}}(\xi_k, \tau_k^-)$, $k = 1, 2, ..., p$, $p \ge 1$.

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