Real Analysis Exchange
Summer Symposium 2010, pp. 55–58

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A NEW CHARACTERIZATION OF THE LORENTZ SPACES L(P,1) FOR $p > 1$ AND APPLICATIONS

The $L(p, 1)$ spaces for $p > 1$ were introduced by G. G. Lorentz in his paper entitled " Some New Functional Spaces" in 1950 in Annals of Mathematics, as the set of all functions $f(x)$, $0 \le x \le 2\pi$ for which

$$\|f\|_{L(p,1)} = \int_0^{2\pi} f^\star(t) t^{\frac{1}{p}-1} dt < \infty$$

where $f^\star$ is the decreasing rearrangement of f.

In this article we show that if $f \in L(p, 1)$ then f can be represented as $f(t) = \sum_{n=1}^\infty c_n d_n(t)$ with $\sum_{n=1}^\infty |c_n| < \infty$, where $d_n(t) = \dfrac{1}{(\mu(A))^{\frac{1}{p}}} \chi_{A_n}(t)$, μ is a measure on subsets of $0 \le x \le 2\pi$, χ_E denotes the characteristic function of the set E, A_n are μ-measurable sets in $0 \le x \le 2\pi$, c_n's are real numbers and $p > 1$.

We denote the space of these representations by $B(\mu, 1/p)$ and endow it with the norm

$$\|f\|_{B(\mu,\{1/p)} = \inf \sum_{n=1}^{\infty} |c_n| \ ,$$

where the infimum is taken over all possible representations of f. Moreover $\|f\|_{L(p,1)}$ is equivalent to $\|f\|_{B(\mu,1/p)}$.

Also we show that if $f \in L(p, 1)$ then f can be represented as $f(t) = \sum_{n=1}^\infty c_n b_n(t)$ with $\sum_{n=1}^\infty |c_n| < \infty$ where $b(t) = \dfrac{1}{\mu[0, 2\pi]}$ or for any $p > 1$ and μ-measurable subsets X, A, B of $0 \le x \le 2\pi$,

$$b(t) = \frac{1}{\mu(X)^{1/p}} \left[\chi_A(t) - \chi_B(t) \right] \ ,$$

Mathematical Reviews subject classification: Primary: 42A99
Key words: Lorentz spaces, special atom spaces, generalized Lipschitz spaces, duality, equivalence of Banach spaces, Besov-Bergman spaces

where $X = A \cup B, A \cap B = \emptyset, \mu(A) = \mu(B)$, μ is a measure on subsets of $0 \le x \le 2\pi$, χ_E denotes the characteristic function of the set E.
We denote the space of these representations by $A(\mu, 1/p)$ and endow it with the norm

$$\|f\|_{A(\mu,1/p)} = \inf \sum_{n=1}^{\infty} |c_n|$$

where the infimum is taken over all possible representations of f. Moreover $\|f\|_{L(p,1)}$ is equivalent to $\|f\|_{A(\mu,1/p)}$.

Consequently, in this paper, we showed that these three spaces $L(p,1)$, $B(\mu, 1/p)$ and $A(\mu, 1/p)$ are equivalent as Banach spaces, and we use these new characterizations of $L(p,1)$ to give a new proof of Carleson's theorem on convergence of Fourier series.

The proof that $B(\mu, 1/p)$ is equivalent to $L(p,1)$ can be done directly and seems to be very simple and straightforward; however, the proof that $B(\mu, 1/p)$ is equivalent to $A(\mu, 1/p)$ and consequently that $L(p,1)$ is equivalent to $A(\mu, 1/p)$ makes use of dualities.

In fact the dual of $B(\mu, 1/p)$ is the set denoted by $Lip(\mu, 1/p)$ of all measurable functions g defined in $0 \le x \le 2\pi$ so that

$$\|f\|_{Lip(\mu,1/p)} = \sup_A \frac{1}{\mu(A)^{1/p}} \left| \int_A f(x) d\mu(x) \right| < \infty \ ,$$

where μ is a measure as in the definition of $B(\mu, 1/p)$ and A is a μ-measurable of $0 \le x \le 2\pi$.

The dual of $A(\mu, 1/p)$ is the set of measurable functions g denoted by $\Lambda(\mu, \alpha)$ defined in $0 \le x \le 2\pi$ so that

$$\|f\|_{\Lambda(\mu,1/p)} = \sup_{X = A\cup B, A\cap B = \emptyset,} \left[\frac{1}{\mu^{1/p}(X)} \left| \int_A f(x) d\mu(x) - \int_B f(x) d\mu(x) \right| \right] < \infty$$

μ-measurable subsets X, A, B of $0 \le x \le 2\pi$.

The space $Lip(\mu, 1/p)$ was introduced by G. G. Lorentz in the same paper mentioned above and the space $\Lambda(\mu, 1/p)$ was introduced by De Souza and Pozo. The authors are not aware of any prior definitions of $\Lambda(\mu, 1/p)$.

APPLICATIONS

One of the immediate applications of this new characterization is to give a simple proof of the classical result due to Guido Weiss-Elias Stein in the 50's that states:

> If T is a sublinear operator on the space of measurable functions and $\|T\chi_A\|_X \le M(\mu(A))^{\frac{1}{p}}, 1 < p < \infty$, where X is a Banach space, then T can be extended to all $L(p,1)$; that is $T : L(p,1) \to X$ and $\|Tf\|_X \le M\|f\|_{L(p,1)}$.

This easily follows from new characterization of $L(p,1)$ as the space $B(\mu, 1/p), 1 < p < \infty$.

A second application is a very important one, since we provide a new proof of Carleson's Theorem on convergence of Fourier series for $L(p,p) = L_p$ by showing it for Lorentz spaces $L(p,1)$ and then using interpolation operators to get it for $L(p,r)$ for $p, r > 1$.

One of the most interesting observations that we obtained in the process to obtain this new characterization of $L(p,1)$ for $1 < p < \infty$ is that $Tf(x) = \operatorname{Sup}_{n\ge1}|S_n(f,x)|$ where is the n^{th} partial sum of the Fourier Series of f is:

> If $Tf(x) = \operatorname{Sup}_{n\ge1}|S_n(f,x)|$, then $\|T\chi_A\|_{L(p,1)} \le M\mu(A)^{\frac{1}{p}}$ for $p > 1$ and so $\|Tf\|_{L(p,1)} \le M\|f\|_{L(p,1)}$, where A is a μ-measurable set.

One very simple proof of this follows easily by Hunt's inequality.

Note: This direct proof using Hunt's inequality was mentioned to the author by Loukas Grafakos during the 23^{th} Mini-Conference on Harmonic Analysis and Related Areas held at Auburn University on December 4-5, 2009 after the talk given by the author on the subject.

Consequently we have Carleson's theorem for $L(p,1)$:

> If $f \in L(p,1)$ then $S_n(f,x)$ converges $f(x)$ almost everywhere.Which is the Carleson's theorem for $L(p,1)$on convergence of Fourier series.

Also we note that $L(p,1) \subseteq L(p,\infty)$ with $\|f\|_{L(p,\infty)} \le C\|f\|_{L(p,1)}$, so it follows by the boundeness of T on $L(p,1)$ that for $p_0 \ne p_1, p_0, p_1 > 1$

a) $\|T\chi_A\|_{L(p_0,\infty)} \le M_0(\mu(A))^{1/p_0}$,

b) $\|T\chi_A\|_{L(p_1,\infty)} \le M_1(\mu(A))^{1/p_1}$.

Therefore, using the interpolation Theorem 1.4.19 in Grafakos, we get

$$\|Tf\|_{L(p,r)} \le M\|f\|_{L(p,r)}, \quad \text{for} \quad \frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, 0 < \theta < 1, \forall r > 1. \qquad (1)$$

The inequality (1) leads to the following, which is a generalization of Carleson's theorem for $L(p,r)$.

If $f \in L(p,r), p, r > 1$, then $S_n(f,x) \to f(x)$ almost everywhere.

If we set $p = r$ we get the classical and well-known Carleson's theorem for $L(p,p) = L_p$, indeed we have,

If $f \in L_p$, then $S_n(f,x) \to f(x)$ almost everywhere.

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