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CHARACTERIZATION OF LACUNARY FUNCTIONS IN WEIGHTED BERGMAN BESOV LIPSCHITZ SPACES

The space B^ρ has been studied at length by various authors for various purposes. This space first appears in its simplest form in [1] where it was denoted by B^1. This was later generalized to a weighted version B^ρ in [2] and [3]. It was shown in these papers that B^ρ is the boundary value of those functions F for which $\int_0^1\int_0^{2\pi}|F'(re^{i\theta})|(1-r)^{\frac{1}{q}-1}d\theta dr<\infty$ for the weight function $\rho(t)=t^{1/q}$. It was shown in [4] that B^ρ, for a general weight function ρ, is a real characterization of analytic functions in the unit disc for which $\int_0^1\int_0^{2\pi}|F'(re^{i\theta})|\frac{\rho(1-r)}{1-r}d\theta dr<\infty$, generalizing the results obtained in [2] and [3].

We consider the weighted Bergman-Besov-Lipschitz space B^ρ of analytic functions F in the unit disc $\mathbb{D}=\{z\in\mathbb{C},|z|\le 1\}$ for which

$$\|F\|_{B^\rho}=\int_0^1\int_0^{2\pi}|F'(re^{i\theta})|\frac{\rho(1-r)}{1-r}d\theta dr$$

and we show that a lacunary function $F(z)=\sum_{n=1}^{\infty}a_nz^n$ belongs to B^ρ if and only if the sequence a_n satisfies $\sum_{n=1}^{\infty}2^nK(n,\rho)\bigg(\sum_{k\in I_n}|a_k|^2\bigg)^{1/2}<\infty$, where I_n are diadic intervals defined by $I_n=\{k\in\mathbb{N}:2^{n-1}\le k<2^n\}$, ρ belongs to a certain class of weights, and $K(n,\rho)>0$ is a function of n and ρ.

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Definition 1. Lacunary functions are analytical functions $F(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ for which $\lambda = \inf_k \frac{n_{k+1}}{n_k} > 1$.

In other words, these are analytic functions that have a natural boundary and thus cannot be continued outside their disk of convergence.

Definition 2. We define

$$B^\rho = \left\{ F : \mathbb{D} \to \mathbb{D}, F \text{ analytic and } \quad \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| \frac{\rho(1-r)}{1-r} d\theta dr < \infty \right\}$$

and

$$b^\rho = \left\{ F : \mathbb{D} \to \mathbb{D}, F(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \sum_{n=0}^{\infty} 2^n K(n,\rho) \left(\sum_{k \in I_n} |a_k|^2 \right)^{1/2} < \infty \right\},$$

where $I_n = \{k \in \mathbb{N} : 2^{n-1} \leq k < 2^n\}$.

Note that B^ρ and b^ρ are endowed respectively with the norms

$$\|F\|_{B^\rho} = \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| \frac{\rho(1-r)}{1-r} d\theta dr$$

and

$$\|F\|_{b^\rho} = \sum_{n=0}^{\infty} 2^n K(n,\rho) \left(\sum_{k \in I_n} |a_k|^2 \right)^{1/2}.$$

Definition 3. We say the weight function $\rho : [0,1] \to [0,\infty)$ belongs to the class $\mathcal{S}$ if $\rho(0) = 0$, ρ is nondecreasing, and there are positive constants $C_1, C_2, K(n,\rho)$ satisfying

$$\int_0^1 r^{2^{n-1}-1} \frac{\rho(1-r)}{1-r} dr \leq C_1 K(n,\rho), \quad \forall n \geq 1 \tag{1}$$

and

$$\int_{1-2^{-(n-1)}}^{1-2^{-n}} r^{2^n-1} \frac{\rho(1-r)}{1-r} dr \geq C_2 K(n,\rho), \quad \forall n \geq 2\ . \tag{2}$$

Hereafter c and C denote generic positive constants and when there is no ambiguity, we shall name all constants by c and C.

Theorem 1. *The class $\mathcal{S}$ is not empty.*

1 Main Theorem

Theorem 2. *Suppose $\rho \in \mathcal{S}$ .Then*

1. *b^ρ is a Banach space.*
2. *For any function $F(z) = \sum_{n=1}^{\infty} a_n z^n$ belonging to B^ρ, there is a constant $C > 0$ such that*
$$\|F\|_{B^\rho} \leq C\|F\|_{b^\rho} .$$
3. *If $F(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is lacunary and belongs to B^ρ, then there is a constant $c > 0$ such that*
$$\|F\|_{B^\rho} \geq c\|F\|_{b^\rho} .$$

Remark 3. *Note that to obtain the upper bound in the main theorem, lacunary sequences are not necessary. The inequality from [5] that allowed us to obtain the lower bounds requires the sequence to be lacunary.*

Remark 4. *In the second part of the proof of the main theorem, our approach of writing the interval $[0, 1]$ as the union of non-overlapping intervals $[1 - 2^{n-1}, 1 - 2^{-n})$, $n \geq 1$, is similar to that in [6].*

To prove the first theorem, we used results about inequalities for the Gamma function in [7].

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