Jun Kawabe,* Department of Mathematics, Shinshu University, 4-17-1 Wakasato, Nagano 3808553, JAPAN. email: `jkawabe@shinshu-u.ac.jp`

NEW SMOOTHNESS CONDITIONS ON RIESZ SPACES WITH APPLICATIONS TO RIESZ SPACE-VALUED NON-ADDITIVE MEASURES AND THEIR CHOQUET INTEGRALS

Abstract

In this summary we introduce a successful analogue of the classical Egoroff theorem for non-additive measures with values in a Riesz space having the asymptotic Egoroff property.

1 Introduction.

In 1974, Sugeno [8] introduced the notion of fuzzy measure and integral to evaluate non-additive or non-linear quality in systems engineering. In the same year, Dobrakov [2] independently introduced the notion of submeasure from mathematical point of view to refine measure theory further. Fuzzy measures and submeasures are both special kinds of non-additive measures, and their studies have stimulated engineers' and mathematicians' interest in non-additive measure theory [1, 7, 9].

When developing non-additive measure theory in a Riesz space, along with the non-additivity of measures, there is a tough technical hurdle to overcome, that is, the ε-argument, which is useful in calculus, does not work in a general Riesz space. Recently, it has been recognized that, as a substitute for the ε-argument, certain smoothness conditions, such as the weak σ-distributivity, the Egoroff property, the asymptotic Egoroff property, and the multiple Egoroff property, should be imposed on a Riesz space to succeed in extending

---

fundamental and important theorems in non-additive measure theory to Riesz space-valued measures. Thus, the study of Riesz space-valued measures will go with such smoothness conditions on the involved Riesz space, and our recent developments reach to non-additive extensions of important theorems in measure theory, such as the Egoroff theorem, the Lebesgue theorem, the Riesz theorem, the Lusin theorem, the Alexandroff theorem, and some convergence theorems of integrals.

In this summary, due to limitations of space, we only introduce a successful analogue of the classical Egoroff theorem for non-additive measures with values in a Riesz space having the asymptotic Egoroff property. See [4] and the bibliography therein for our other contributions to Riesz space-valued non-additive measure theory.

2   The Egoroff theorem.

In what follows, we always assume that V is a Riesz space and $(X, \mathcal{F})$ is a measurable space. Denote by $\mathbb{N}$ the set of all natural numbers. A set function $\mu : \mathcal{F} \to V$ is called a *non-additive measure* if $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{F}$ and $A \subset B$.

The classical theorem of Egoroff is one of the most fundamental and important theorems in measure theory. This asserts that almost everywhere convergence implies almost uniform convergence (and hence convergence in measure) and gives a key to handle a sequence of measurable functions. However, it is known that the Egoroff theorem does not hold in general for non-additive measures.

Recently, Murofushi *et al.* [6] discovered a necessary and sufficient condition, called the Egoroff condition, which assures that the Egoroff theorem is still valid for real valued non-additive measures, and indicated that the continuity of a non-additive measure is one of the sufficient conditions for the Egoroff condition; see also Li [5]. The Egoroff condition can be naturally described for Riesz space-valued non-additive measures. Denote by Θ the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$.

Definition 1 ([3])**.** Let $\mu : \mathcal{F} \to V$ be a non-additive measure.
1. A double sequence $\{A_{m,n}\}_{(m,n) \in \mathbb{N}^2} \subset \mathcal{F}$ is called a *μ-regulator* in $\mathcal{F}$ if it satisfies the following two conditions:

   (i) $A_{m,n} \supset A_{m,n'}$ whenever $m, n, n' \in \mathbb{N}$ and $n \leq n'$.

   (ii) $\mu\left(\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty} A_{m,n}\right) = 0$.

2. We say that μ satisfies *the Egoroff condition* if $\inf_{\theta \in \Theta} \mu\left(\bigcup_{m=1}^{\infty} A_{m,\theta(m)}\right) = 0$ for any μ-regulator $\{A_{m,n}\}_{(m,n) \in \mathbb{N}^2}$ in $\mathcal{F}$.

Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of $\mathcal{F}$-measurable, real-valued functions on X and f also such a function. Recall that $\{f_n\}_{n \in \mathbb{N}}$ converges μ-*almost everywhere* to f if there is a set $E \in \mathcal{F}$ with $\mu(E) = 0$ such that $f_n(x)$ converges to $f(x)$ for all $x \in X - E$, and converges μ-*almost uniformly* to f if there is a decreasing net $\{E_\alpha\}_{\alpha \in \Gamma} \subset \mathcal{F}$ with $\mu(E_\alpha) \downarrow 0$ such that f_n converges to f uniformly on each set $X - E_\alpha$. The following gives a Riesz space version of [6, Proposition 1].

Theorem 1 ([3]). *Let $\mu : \mathcal{F} \to V$ be a non-additive measure. Then, μ satisfies the Egoroff condition if and only if the Egoroff theorem holds for μ, that is, for any sequence $\{f_n\}_{n \in \mathbb{N}}$ of $\mathcal{F}$-measurable, real-valued functions on X converging μ-almost everywhere to such a function f on X, it converges μ-almost uniformly to the same limit f.*

Li [5, Theorem 1] proved that the Egoroff theorem holds for any continuous real-valued non-additive measure. Its proof is essentially based on the ε-argument which does not work in a general Riesz space. Therefore, it seems that, as a substitute for the ε-argument, some smoothness conditions should be introduced and imposed on a Riesz space to obtain a successful analogue of the Egoroff theorem for Riesz space-valued non-additive measures. The following is one of our new smoothness conditions on a Riesz space by which we will develop Riesz space-valued non-additive measure theory.

Definition 2 ([3]). Let $u \in V^+$. For each $m \in \mathbb{N}$, consider a multiple sequence $u^{(m)} := \{u_{n_1,\ldots,n_m}\}_{(n_1,\ldots,n_m) \in \mathbb{N}^m}$ of elements of V.
1. A sequence $\{u^{(m)}\}_{m \in \mathbb{N}}$ of the multiple sequences is called a u-*multiple regulator* in V if, for each $m \in \mathbb{N}$ and $(n_1, \ldots, n_m) \in \mathbb{N}^m$, the multiple sequence $u^{(m)}$ satisfies the following two conditions:

   (i) $0 \le u_{n_1} \le u_{n_1,n_2} \le \cdots \le u_{n_1,\ldots,n_m} \le u$.

   (ii) Letting $n \to \infty$, then $u_n \downarrow 0$, $u_{n_1,n} \downarrow u_{n_1}$, ..., and $u_{n_1,\ldots,n_m,n} \downarrow u_{n_1,\ldots,n_m}$.

2. We say that V has *the asymptotic Egoroff property* if, for each $u \in V^+$ and each u-multiple regulator $\{u^{(m)}\}_{m \in \mathbb{N}}$, the following two conditions hold:

   (i) $u_\theta := \sup_{m \in \mathbb{N}} u_{\theta(1),\ldots,\theta(m)}$ exists for each $\theta \in \Theta$.

   (ii) $\inf_{\theta \in \Theta} u_\theta = 0$.

We are now ready to give a Riesz space version of [5, Theorem 1].

Theorem 2 ([3]). *Let $\mu : \mathcal{F} \to V$ be a non-additive measure. Assume that V has the asymptotic Egoroff property. Then, μ satisfies the Egoroff condition whenever it is continuous, that is, $\mu(A_n) \to \mu(A)$ whenever a sequence $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ monotonically converges to $A \in \mathcal{F}$.*

3   Conclusion.

We give a very short summary of our recent developments in Riesz space-valued non-additive measure theory. Such a study goes with smoothness conditions on the involved Riesz space, because the ε-argument, which is useful in the existing theory of measures, does not work well in a general Riesz space. Typical examples of Riesz spaces satisfying our smoothness conditions are the Lebesgue function spaces $\mathcal{L}_p[0,1]$ $(0 < p \le \infty)$, so that the established results could be instrumental when developing non-additive extension of the theory of p-th order stochastic processes and fuzzy number-valued measure theory.

References

[1] D. Denneberg, *Non-Additive Measure and Integral*, Second edition, Kluwer Academic Publishers, Dordrecht, 1997.

[2] I. Dobrakov, *On submeasures I*, Dissertationes Math. (Rozprawy Mat.) **112** (1974), 1–35.

[3] J. Kawabe, *The Egoroff theorem for non-additive measures in Riesz spaces*, Fuzzy Sets and Systems **157** (2006), 2762–2770.

[4] J. Kawabe, *A study of Riesz space-valued non-additive measures*, in: V.-N. Huynh *et al.* (Eds.), Integrated Uncertainty Management and Applications, Advances in Intelligent and Soft Computing **68** (2010), 91–102.

[5] J. Li, *Order continuous of monotone set function and convergence of measurable functions sequence*, Appl. Math. Comput. **135** (2003), 211–218.

[6] T. Murofushi, K. Uchino, and S. Asahina, *Conditions for Egoroff's theorem in non-additive measure theory*, Fuzzy Sets and Systems **146** (2004), 135–146.

[7] E. Pap, *Null-Additive Set Functions*, Kluwer Academic Publishers, Dordrecht, 1995.

[8] M. Sugeno, *Theory of Fuzzy Integrals and Its Applications*, Thesis, Tokyo Institute of Technology, 1974.

[9] Z. Wang and G. J. Klir, *Generalized Measure Theory*, Springer, New York, 2009.