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UNIFORMIZABLE SUBTOPOLOGY AND A SORT OF CONVERSE OF A. H. STONE'S THEOREM

Abstract

Heuristically, the stunning feature of the real line i.e. the set of reals R with the usual topology $\mathcal{U}$ is uniformizable whereas its countable complement extension topology is not, stems the problem – whether, more generally, a topological space, uniformizable or not, has a non-trivial proper uniformizable subtopology (other than the subtopology $\{\emptyset, X, U, V\}$, which is always uniformizable, for a disconnection $\{U, V\}$ of X, in the case when X is disconnected). In this paper, sufficient conditions are given for spaces to have non-trivial proper uniformizable subtopologies, where a topology σ on X is called a subtopology of a space (X, τ) if $\sigma \subset \tau$. An useful consequence of this investigation reflects that a sort of converse of the famous A. H. Stone's theorem is true. In this study, disconnectedness plays a major role, specially when it is of very strong in nature like zero dimensionality; then, for a paracompact T_2 space containing no isolated points, the cardinality of such subtopologies is at least $\aleph_0$. It has also been established as to when a space does not possesses any such subtopology.

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1 Preliminaries.

By X, we shall mean a topological space without any separation axioms. For a cover $\mathcal{U}$ of X and for a subset A of X, the star of A with respect to $\mathcal{U}$ is the set $St(A;\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$; and for two covers $\mathcal{U}$ and $\mathcal{V}$ of X, we call $\mathcal{U}$ *star refines* $\mathcal{V}$ or $\mathcal{U}$ is a *star refinement* of $\mathcal{V}$, written $\mathcal{U} \overset{\star}{<} \mathcal{V}$, if for each $U \in \mathcal{U}$, $\exists\ V \in \mathcal{V}$ such that $St(U;\mathcal{U}) \subset V$. $\mathcal{U}$ is called a *refinement* of $\mathcal{V}$ written as $\mathcal{U} < \mathcal{V}$ if for each $U \in \mathcal{U}$, there exists $V \in \mathcal{V}$ such that $U \subset V$.

A *normal sequence* of covers is a sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, ...$ of X such that $\mathcal{U}_{n+1} \overset{\star}{<} \mathcal{U}_n$ for $n = 1, 2, ...$; and a *normal cover* is cover which is $\mathcal{U}_1$ in some normal sequence of covers.

A collection μ' of covers of a space X is a *base* for some uniformity on X iff it satisfies the condition that for $\mathcal{U}_1, \mathcal{U}_2 \in \mu'$ there is some $\mathcal{U}_3 \in \mu'$ such that $\mathcal{U}_3 \overset{\star}{<} \mathcal{U}_1$ and $\mathcal{U}_3 \overset{\star}{<} \mathcal{U}_2$ [§ 36.3, Page-245 [4]]. It is well known that if μ' is a base for a covering uniformity μ on X, then $\{St(x;\mathcal{U}) : \mathcal{U} \in \mu'\}$ is a nbd. base at $x \in X$ in the uniform topology [§36.6, Page 246 [4]]. Also if X is any uniformizable topological space then there is a finest uniformity on X, compatible with the topology of X, called the *fine uniformity* on X, denoted by μ_F, having a base of all normally open covers of X ; such a space is known as *fine space*. Further, a T_1 space is paracompact iff every open cover has an open star refinement [§20.14, Page-149 [4]], where a space X is called *paracompact* iff every open cover of X has an open locally finite refinement, which is also a cover of X.

2 Uniformizable Subtopology.

Since one can check that μ_0 , the collection of all normally open covers of (X, τ), is a normal family, then by Theorem 36.11, Page-248, [4], μ_0 is a subbase for some uniformity μ on X. Let μ' be the base for μ obtained from the subbase μ_0 and τ' be the corresponding uniform topology . Now, $\beta'_x = \{St(x;\mathcal{U}) : \mathcal{U} \in \mu'\}$ is a nbd. base at $x \in X$ in (X, τ'). Then $\beta'_x \subset \beta_x$, for all $x \in X$ where β_x is a nbd. base at x in (X, τ). If τ is not uniformizable then as τ' is being uniformizable , $\tau' \subsetneqq \tau$. That τ' is non trivial if (X, τ) is disconnected; in fact, for a disconnection (U, V) of (X, τ), $\mathcal{U} = \{U, V\}$ is an open cover of (X, τ) and also is a partition of X. Therefore, $\mathcal{U}$ star refines itself and hence is a normally open cover and so $\mathcal{U} \in \mu_0$. Consequently $\mathcal{U} \in \mu'$. Here $St(x;\mathcal{U})$ is either U if $x \in U$ or V if $x \in V$.

Clearly, U $\in \beta'_x$, $\forall x \in U$. So U contains a basic nbd. of each of its points in (X, τ') and hence $U \in \tau'$. Since $U \neq X, \emptyset$, τ' is therefore nontrivial. Hence we have the following result:

Theorem 1. *If (X, τ) is a disconnected non uniformizable topological space containing at least three points then there exists a non trivial proper uniformizable subtopology on X.*

Remark 1. The assumptions of disconnectedness and $card(X) \geq 3$ are essential.

Example 1. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a, b\}, \{d\}, \{a, b, d\}\}$. Then (X, τ) is a non-discrete, non-uniformizable, connected space having no non trivial strictly smaller uniformizable subtopology. In fact, all four non trivial proper subtopologies are non uniformizable.

Example 2. Let $X = \{a, b\}$. Then the collection of all topologies on X is $\{\tau_1 = \{\emptyset, X, \{a\}\}, \tau_2 = \{\emptyset, X, \{b\}\}, \tau_3 =$discrete, τ_4=indiscrete$\}$. Here τ_1, τ_2 are connected , non-uniformizable, non discrete, nontrivial topologies on X, both of which have only one strictly smaller topology which is indiscrete.
That the assumption of disconnectedness is not necessary, follows from the next example:

Example 3. Let τ be the countable complement extension topology of the real line with the usual topology $(R,\mathcal{U})$. Then obviously (R, τ) is a non-uniformizable, connected space and has the subtopology which is the usual topology $\mathcal{U}$, that is non trivial, proper as well as uniformizable.

It is not hard to prove the following theorem :

Theorem 2. *Let (X, τ) be a non-uniformizable non-compact space (without T_1-axiom) in which every open cover has an open star refinement, Then (X, τ) has a proper nontrivial uniformizable subtopology.*

On the other hand, suppose (X, τ) is paracompact T_2 but non metrizable,then every open cover is a normally open cover and also as (X, τ) is being T_2, for all $x \neq y \in X$, there exist disjoint open sets O_x and O_y containing x and y respectively. Clearly O_x and O_y are proper open subsets of X, for all $x \neq y \in X$. Fix y_0, $x_0 \in X$ and take $\mathcal{U} = \{O_x, x(\neq x_0) \in X$ such that $O_{x_0} \cap O_x = \emptyset, x_0 \in O_{x_0}, x \in O_x, O_{x_0}, O_x \in \tau\} \cup \{O_{x_0} \in \tau : O_{x_0} \cap O_{y_0} = \emptyset, x_0 \in O_{x_0}, y_0 \in O_{y_0}, O_{y_0} \in \tau\}$; then $\mathcal{U}$ is an τ-open cover of X. So, $\mathcal{U}$ is a normally cover. Let $\mathcal{U} = \mathcal{U}_1,\mathcal{U}_2$,.... be the corresponding normal sequence of open covers. Then $\mu' = \{\mathcal{U}_1,\mathcal{U}_2, ...\}$ is a base for some uniformity μ on X. Let τ_μ be the corresponding uniform topology induced by the uniformity μ on X. Then τ_μ is pseudometrizable as μ has a countable base μ'. The family $\beta_x = \{St(x;\mathcal{U}_i) : i = 1, 2, 3, ...\}$ is a nbd. base at $x \in X$ in (X, τ_μ). Here it is to be noted that all members of all $\mathcal{U}_i$'s are proper open subsets of X . Now for $St(x;\mathcal{U}_i) \in \beta_x$, $x \in X$, $St(x;\mathcal{U}_i) \subseteq St(U;\mathcal{U}_i)$ for some $U \in \mathcal{U}_i$ such that $x \in U$. Also $St(U;\mathcal{U}_i) \subset V$ for some $V \in \mathcal{U}_{i-1}$ and as $V \subsetneq X$, then $St(x;\mathcal{U}_i) \subset V \subsetneq X$. So, β_x contains many proper subsets of X. Therefore τ_μ contains proper nonempty open sets, and hence τ_μ is a non-trivial topology on X.

Now if μ'_1 be the collection of all open covers of X then μ'_1 is the base for the fine uniformity μ_F on X [as (X,τ) is paracompact T_2, so every open cover of (X,τ) is normally open cover, and as the family of all normally open covers of a uniformizable space (X,τ), forms base for the fine uniformity μ_F on X, which induces the topology τ] which gives the topology τ as the uniform topology. Now $\mu' \subset \mu'_1 \Rightarrow \tau_\mu \subseteq \tau = \tau_{\mu_F}$. As τ is not pseudometrizable and τ_μ is pseudometrizable, so $\tau_\mu \subsetneq \tau$. Hence X has a proper non-trivial subtopology τ_μ, which is uniformizable [as generated by the uniformity μ]. Hence we have the following theorem :

Theorem 3. *A non-metrizable, paracompact T_2 space X has a proper, non-trivial uniformizable subtopology.*

The famous A.H. Stone's theorem states that every metrizable space is paracompact T_2. Whereas βN, the Stone Čech compactification of the set of naturals N (with the discrete topology) is paracompact T_2 but is not metrizable. But we derive the following converse i.e. Corollary 1:

Indeed, in the discussion before Theorem 3, we had the subtopology τ_μ, which was come from a uniformity μ with a countable base μ'. So, τ_μ is pseudometrizable such that $\tau_{ind.} \subsetneq \tau_\mu \subseteq \tau$. If $\tau_\mu = \tau$ then as (X,τ) is T_2, (X,τ) is metrizable. If $\tau_\mu \subsetneq \tau$ then τ_μ is a proper non trivial pseudometrizable subtopology of τ. Hence we have the corollary:

Corollary 1. *If (X,τ) is paracompact T_2 then either (X,τ) is metrizable or (X,τ) has a proper nontrivial subtopology which is pseudometrizable.*

Remark 2. In Theorem 3, 'paracompact-ness' is not necessary.

Example 4. Let $Y = \beta N$, where βN is the Stone-Čech compactification of the set of natural numbers N and consider the topology τ on Y generated by the topology τ' of βN together with the sets $N \cup \{y\}$, for all $y \in \beta N - N$. Clearly (Y,τ) is non-compact T_2-space. Also (Y,τ) is non-completely regular. In fact it is non-regular. Consider the closed set $\beta N - (N \cup \{y\})$ and $y \notin \beta N - (N \cup \{y\})$. Since any open set containing y in (Y,τ) is either $N \cup \{y\}$ or meeting N and as $\overline{N}$ (in (Y,τ')) $= \beta N$, then every open set in Y containing $\beta N - (N \cup \{y\})$ meets N. Hence (Y,τ) is also non-paracompact, non-metrizable as well as non uniformizable. But the non-trivial topology τ' on βN is strictly weaker than the topology τ on $Y = \beta N$ and $(\beta N,\tau')$ is compact T_2 and hence is uniformizable.

In case (X,τ) is disconnected T_2 containing no isolated points, then it has a disconnection(U,V) of X; obviously both U and V contains infinite number of points. Let $x_0, y_0, x_1, y_1, ..., x_n, y_n \in V$ and all are distinct. Now as (X,τ) is T_2, we get two disjoint open sets $O_{x_i}(y)$ and $O_y(x_i)$ containing y and x_i respectively, two disjoint open sets $O_{y_i}(y)$ and $O_y(y_i)$ containing y and y_i respectively and another two disjoint open sets $O'_{y_i}(x_i)$ and $O'_{x_i}(y_i)$ containing x_i and y_i respectively for each $i \in \{0, 1, ..., n\}$ and for each $y(\neq x_i, y_i : i = 0, 1, 2, ..., n) \in V$. Instead of $O'_{y_0}(x_0)$, $O'_{x_0}(y_0)$, ..., $O'_{y_n}(x_n)$, $O'_{x_n}(y_n)$ we take open sets $O_{y_0}(x_0), O_{x_0}(y_0), ..., O_{y_n}(x_n)$, $O_{x_n}(y_n)$ which are pairwise disjoint. Such sets are constructed in the following way:
By T_2 property for the two points x, y $(x \neq y) \in X$, we have two disjoint open sets $U_x(y)$ and $U_y(x)$ containing y and x respectively. Now consider the following chart.

For x_0 and x_i, $i = 1, 2, .., n$ and for x_o, y_i, $i = 0, 1, 2, .., n$	For x_1 and x_i, $i = 2, 3, .., n$ and for x_1 and y_i, $i = 0, 1, 2, .., n$	..	For x_{n-1} and x_n, and x_{n-1}, y_i $i = 0, 1, 2, .., n$	For y_0 and y_i, $i = 1, 2, .., n$	For y_1 and y_n, $i = 2, 3, .., n$	..	For y_{n-1} and y_n
$U_{x_1}(x_0)$, $U_{x_0}(x_1)$;							
$U_{x_2}(x_0)$, $U_{x_0}(x_2)$;	$U_{x_2}(x_1)$, $U_{x_1}(x_2)$;						
$U_{x_3}(x_0)$, $U_{x_0}(x_3)$;	$U_{x_3}(x_1)$, $U_{x_1}(x_3)$;						
$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$					
$U_{x_n}(x_0)$, $U_{x_0}(x_n)$;	$U_{x_n}(x_1)$, $U_{x_1}(x_n)$;	..	$U_{x_n}(x_{n-1})$, $U_{x_{n-1}}(x_n)$;				
$U_{y_0}(x_0)$, $U_{x_0}(y_0)$;	$U_{y_0}(x_1)$, $U_{x_1}(y_0)$;	..	$U_{y_0}(x_{n-1})$, $U_{x_{n-1}}(y_0)$;				
$U_{y_1}(x_0)$, $U_{x_0}(y_1)$;	$U_{y_1}(x_1)$, $U_{x_1}(y_1)$;	..	$U_{y_1}(x_{n-1})$, $U_{x_{n-1}}(y_1)$;	$U_{y_1}(y_0)$, $U_{y_0}(y_1)$;			
$U_{y_2}(x_0)$, $U_{x_0}(y_2)$;	$U_{y_2}(x_1)$, $U_{x_1}(y_2)$;	..	$U_{y_2}(x_{n-1})$, $U_{x_{n-1}}(y_2)$;	$U_{y_2}(y_0)$, $U_{y_0}(y_2)$;	$U_{y_2}(y_1)$, $U_{y_1}(y_2)$;		
$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$	
$U_{y_n}(x_0)$, $U_{x_0}(y_n)$;	$U_{y_n}(x_1)$, $U_{x_1}(y_n)$;	..	$U_{y_n}(x_{n-1})$, $U_{x_{n-1}}(y_n)$;	$U_{y_n}(y_0)$, $U_{y_0}(y_n)$;	$U_{y_n}(y_1)$, $U_{y_1}(y_n)$;	..	$U_{y_n}(y_{n-1})$, $U_{y_{n-1}}(y_n)$;

Now considering the intersection of $O'_{y_i}(x_i)$ and all open sets containing x_i in the above chart, we shall get $O_{y_i}(x_i)$, $i = 0, 1, 2, \ldots, n$. Similarly, the intersection of $O'_{x_i}(y_i)$ and all open sets containing y_i in the above chart we shall get $O_{x_i}(y_i)$, $i = 0, 1, 2, \ldots, n$. Clearly $O_{x_0}(y_0)$, $O_{y_0}(x_0)$, $\ldots$, $O_{x_n}(y_n)$, $O_{y_n}(x_n)$ are pairwise disjoint open sets. Let $\mathcal{U}_V = \{O_{x_0}(y) \cap O_{y_0}(y) \cap \ldots \cap O_{x_n}(y) \cap O_{y_n}(y) \cap V : y \neq x_i, y_i, i = 0, 1, 2, \ldots, n\} \cup \{O_{y_0}(x_0) \cap V, O_{x_0}(y_0) \cap V, \ldots, O_{y_n}(x_n) \cap V, O_{x_n}(y_n) \cap V\}$. Here sets $O_y(x_0) \cap V, O_y(y_0) \cap V, \ldots, O_y(x_n) \cap V, O_y(y_n) \cap V$ for $y(\neq x_i, y_i, i = 0, 1, 2, \ldots, n) \in V$ are not taken. Let $\mathcal{U} = \{U\} \cup \mathcal{U}_V$. Here $\mathcal{U}$ is a cover of X containing proper open subsets of (X, τ) and every member of $\mathcal{U}_V$ is contained in V.

If in addition (X, τ) is paracompact then $\mathcal{U}$ is normally open cover and so there exists a normal sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ with $\mathcal{U} = \mathcal{U}_1$. Here $\mathcal{U}_k \overset{\star}{<} \mathcal{U}_1 = \mathcal{U}$, $\forall k > 1$ and any $W \in \mathcal{U}_k$ is either a subset of U or contained in some member of $\mathcal{U}_V$, as U is disjoint with each member of $\mathcal{U}_V$.

We shall consider a new sequence of open covers $\mathcal{U}_1^{'}, \mathcal{U}_2^{'}, \mathcal{U}_3^{'}, \ldots$, where $\mathcal{U} = \mathcal{U}_1^{'} = \mathcal{U}_1$, and for $k > 1$, $\mathcal{U}_k^{'}$ is $\mathcal{U}_k$, when $\mathcal{U}_k$ contains only U and no other subsets of U, and if $\mathcal{U}_k$ contains at least one proper open subset of (X, τ) which is contained in U then we remove all these proper subsets of U and replace them by U if U is not in $\mathcal{U}_k$ and get $\mathcal{U}_k^{'}$. So it is easy to see that $\ldots \overset{\star}{<} \mathcal{U}_3^{'} \overset{\star}{<} \mathcal{U}_2^{'} \overset{\star}{<} \mathcal{U}_1^{'} = \mathcal{U}$.

So, $\mathcal{U}$ is a normally open cover of (X, τ) . Now we see that $\mu = \{\mathcal{U}_1^{'}, \mathcal{U}_2^{'}, \ldots\}$ is clearly a base for some uniformity $\mu^{'}$ on X. Let $\tau_{\mu'}$ be the topology induced by $\mu^{'}$ on X. Let μ_1 be the collection of all normally open covers of (X, τ). Then μ_1 is the base for the fine uniformity $\mu_1^{'}$ on X, which induces the topology τ and $\mu \subseteq \mu_1$. So, $\tau_{\mu'} \subseteq \tau_{\mu_1'}$, where $\tau_{\mu_1'}$ is the uniform topology induced by the fine uniformity $\mu_1^{'}$ on X. As $\tau_{\mu_1'} = \tau$, so $\tau_{\mu'} \subseteq \tau$. Since the family $\beta_x = \{St(x; \mathcal{U}_k^{'}) : k = 1, 2, \ldots\}$ forms a nbd. base at $x \in X$ for $(X, \tau_{\mu'})$ and $St(x; \mathcal{U}_k^{'}) = U$ for all $x \in U$ and any k, so U is open in $(X, \tau_{\mu'})$. Therefore, $\tau_{ind} \subsetneqq \tau_{\mu'}$.

Next shall show that $\tau_{\mu'} \subsetneqq \tau$. In fact,for $x_0^{'}, y_0^{'} \in U$, as $\forall\ x \in U \cap O_{y_0'}(x_0^{'})$, $U \cap O_{y_0'}(x_0^{'}) \subsetneqq U = St(x; \mathcal{U}_k^{'})$, $\forall k$ and $\forall y \in U \cap O_{x_0'}(y_0^{'})$, $U \cap O_{x_0'}(y_0^{'}) \subsetneqq U = St(y; \mathcal{U}_k^{'})$, $\forall k$, then for any $x \in U \cap O_{y_0'}(x_0^{'})$, $U \cap O_{y_0'}(x_0^{'})$, does not contain any member of the nbd. base β_x at x in $(X, \tau_{\mu'})$. So, $U \cap O_{y_0'}(x_0^{'}) \notin \tau_{\mu'}$. Similarly $U \cap O_{x_0'}(y_0^{'}) \notin \tau_{\mu'}$. But they are open in (X, τ).

It can also be shown that $\tau_{ind} \subsetneqq \tau_T \subsetneqq \tau$ where $\tau_T = \{\emptyset, X, U, V\}$, τ_T is obviously also uniformizable. Hence we have the following theorem:

Theorem 4. *If (X,τ) is a paracompact T_2 disconnected space containing no isolated points then (X,τ) has a non-trivial, proper uniformizable subtopology (different from $\tau_T = \{\emptyset, X, U, V\}$ which comes from any disconnection $\{U,V\}$ of X).*

Let $\mathcal{U}_V$,$O_{x_i}(y_i)$ and $O_{y_i}(x_i)$ be as before. If the disconnectedness of the above Theorem is seen zero-dimensionality, there exist clopen sets $U^{'}{}_{x_i}(y_i)$, $U^{'}{}_{y_i}(x_i)$ such that $y_i \in U^{'}{}_{x_i}(y_i) \subset O_{x_i}(y_i)$ and $x_i \in U^{'}{}_{y_i}(x_i) \subset O_{y_i}(x_i)$. Consider the open cover $\mathcal{V} = \{H_{x_i}(y_i), H_{y_i}(x_i) : i = 0,1,2,\ldots,n\} \cup \{U\} \cup \{V \cap (H_{x_0}(y_0) \cup H_{y_0}(x_0) \cup \ldots \cup H_{y_n}(x_n) \cup H_{x_n}(y_n))^c\}$, where $U^{'}{}_{x_i}(y_i) \cap V = H_{x_i}(y_i)$ and $U^{'}{}_{y_i}(x_i) \cap V = H_{y_i}(x_i)$. Here elements of $\{H_{x_i}(y_i) : i = 0,1,2,\ldots,n, H_{y_i}(x_i) : i = 0,1,2,\ldots,n\}$ are pairwise disjoint and choose an arbitrary element of say $H_{x_k}(y_k)$, and keep it fixed. For two distinct points a, b of $H_{x_k}(y_k)$, applying T_2-property for z and a $(z \neq a,b)$, $z \in H_{x_k}(y_k)$ we get two disjoint open sets $O_z(a)$ and $O_a(z)$ containing a and z respectively. We also get two disjoint open sets $O_a(b)$ and $O_b(a)$ containing b and a respectively. Now consider the open cover $\mathcal{V}_{H_{x_k}(y_k)}$ of X as $\mathcal{V}_{H_{x_k}(y_k)} = \{H_{x_k}(y_k) \cap O_b(a)\} \cup \{H_{x_k}(y_k) \cap O_a(z) : z(\neq a) \in H_{x_k}(y_k)\} \cup \{U\} \cup \{H_{x_i}(y_i) : i = 0,1,\ldots,k-1,k+1,\ldots,n\} \cup \{H_{y_i}(x_i) : i = 0,1,\ldots,n\} \cup [V \cap \{H_{x_0}(y_0) \cup H_{y_0}(x_0) \cup \ldots \cup H_{y_n}(x_n) \cup H_{x_n}(y_n)\}^c]$. Then the only member of $\mathcal{V}_{H_{x_k}(y_k)}$ which contains the point a is $H_{x_k}(y_k) \cap O_b(a)$.

If in addition (X,τ) is paracompact, this cover is a normally open cover of X. So, there exists a normal sequence of open covers $(\mathcal{W}_1^{H_{x_k}(y_k)})', (\mathcal{W}_2^{H_{x_k}(y_k)})', \ldots$ such that $(\mathcal{W}_1^{H_{x_k}(y_k)})' = \mathcal{V}_{H_{x_k}(y_k)}$. Now consider one element of $\{H_{x_i}(y_i) : i = 0,1,\ldots,k-1,k+1,\ldots,n\} \cup \{H_{y_i}(x_i) : i = 0,1,\ldots,n\}$ [which are members of $\mathcal{V}_{H_{x_k}(y_k)}$] say $H_{y_l}(x_l)$ and take an open cover $(\mathcal{W}_r^{H_{x_k}(y_k)})'$ of the above normal sequence of open covers. If $(\mathcal{W}_r^{H_{x_k}(y_k)})'$ does not contain any subset of $H_{y_l}(x_l)$, then $H_{y_l}(x_l)$ must belongs to $(\mathcal{W}_r^{H_{x_k}(y_k)})'$ and in this case we keep $(\mathcal{W}_r^{H_{x_k}(y_k)})'$ unchanged, but if $(\mathcal{W}_r^{H_{x_k}(y_k)})'$ contains at least one subset of $H_{y_l}(x_l)$ then two cases arises:

Case-I: $H_{y_l}(x_l) \in (\mathcal{W}_r^{H_{x_k}(y_k)})'$. Case-II: $H_{y_l}(x_l) \notin (\mathcal{W}_r^{H_{x_k}(y_k)})'$.

In Case-I, we delete all subsets of $H_{y_l}(x_l)$ except $H_{y_l}(x_l)$ itself from $(\mathcal{W}_r^{H_{x_k}(y_k)})'$. For Case-II, we delete all subsets of $H_{y_l}(x_l)$ from $(\mathcal{W}_r^{H_{x_k}(y_k)})'$ and take $H_{y_l}(x_l)$ in $(\mathcal{W}_r^{H_{x_k}(y_k)})'$. We do this for each of $(\mathcal{W}_1^{H_{x_k}(y_k)})'$, $(\mathcal{W}_2^{H_{x_k}(y_k)})'$, … and for each element of $\{H_{x_i}(y_i) : i = 0,1,\ldots,k-1,k+1,\ldots,n\} \cup \{H_{y_i}(x_i) : i = 0,1,\ldots,n\}$ and get a new sequence of open covers say $(\mathcal{W}_1^{H_{x_k}(y_k)})$, $(\mathcal{W}_2^{H_{x_k}(y_k)})$, … which is again a normal sequence of open covers of X. Here clearly $\mathcal{W}_1^{H_{x_k}(y_k)} =$

$\mathcal{V}_{H_{x_k}(y_k)}$. Let $\mu'_{H_{x_k}(y_k)} = \{\mathcal{W}_1^{H_{x_k}(y_k)}, \mathcal{W}_2^{H_{x_k}(y_k)}, \ldots\}$, then $\mu'_{H_{x_k}(y_k)}$ forms a base for some uniformity, say $\mu_{H_{x_k}(y_k)}$ on X, and because of the discussion before Theorem 4, which induces a uniform topology $\tau_{\mu_{H_{x_k}(y_k)}} \in \mathcal{F}$, the family of all proper nontrivial uniformizable subtopologies of τ . So we get $\tau_{\mu_{H_{x_0}(y_0)}}, \tau_{\mu_{H_{x_1}(y_1)}}, \ldots, \tau_{\mu_{H_{x_n}(y_n)}}, \tau_{\mu_{H_{y_0}(x_0)}} \ldots \tau_{\mu_{H_{y_n}(x_n)}}$ i.e. $2(n+1) = 2(n+1)_{C_1}$ topologies which are clearly members of $\mathcal{F}$. One can check that all these $2(n+1)$ topologies are different. Now if we start with taking two arbitrary elements of $\{H_{y_i}(x_i) : i = 0, 1, \ldots, n\} \cup \{H_{x_i}(y_i) : i = 0, 1, \ldots, n\}$ instead of one and do the same procedure we get $2(n+1)_{C_2}$ different topologies of the form $\tau_{\mu_{\{H_{x_0}(y_0), H_{x_1}(y_1)\}}}, \tau_{\mu_{\{H_{x_0}(y_0), H_{x_2}(y_2)\}}} \ldots$ etc. which are all distinct members of $\mathcal{F}$. Also if we start with the open cover $\mathcal{V}$ of X then we get a topology $\tau_{\mathcal{V}}$ in $\mathcal{F}$. So we get $2(n+1)_{C_0} + 2(n+1)_{C_1} + \ldots + 2(n+1)_{C_{2(n+1)}} = 2^{2(n+1)}$ distinct topologies in $\mathcal{F}$. Hence by the mathematical induction, we have:

Theorem 5. *If (X, τ) be a paracompact (T_2) zero-dimensional space containing no isolated points and if $\mathcal{F}$ be the family of all proper nontrivial uniformizable subtopologies of τ then $|\mathcal{F}| \geq \aleph_0$.*

Remark 3. The cardinality of non-trivial proper uniformizable subtopologies of the Cantor space is at least $\aleph_0$. Now using the next theorem, we establish Theorem 7:

Theorem 6 ([1])**.** *A topological space (X, τ) is disconnected iff it has an open cover $\mathcal{U}$ consisting of proper subsets of X such that $\mathcal{U} \overset{\star}{<} \mathcal{U}$.*

Theorem 7. *If (X, τ) is non uniformizable, connected with $card(X) > \aleph_0$ and $card(\tau)$ is finite then there is no proper non-trivial uniformizable subtopology of (X, τ).*

Remark 4. The condition that $card(\tau)$ is finite can not be dropped. In Example 3, that (R, τ) is a non-uniformizable connected space with $card(R) > \aleph_0$ and $card(\tau)$ is infinite; but it has a non-trivial proper uniformizable subtopology viz. the usual topology $\mathcal{U}$.

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