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# ON LUSIN-TYPE APPROXIMATELY CONTINUOUS INTEGRALS

The notion of approximate derivative has its origin in the work of Lusin [8]. It was born in order to serve in a descriptive definition of wide Denjoy integral. Despite the Lusin's definition of wide Denjoy integral uses approximate differentiation, not all *exact*[1] approximate derivatives are Denjoy integrable (only those with continuous primitives: simply because Denjoy integral is by definition a continuous integral). Soon, however, several integration processes that allow to recover a function from its exact approximate derivative, appeared. Among the most common that cover also Lebesgue integration are approximately continuous Burkill's integral [1] and approximately continuous Kurzweil–Henstock integral. Both the latter are in fact special cases of some more general processes defined with respect to a derivation base.

The connection between the aforementioned approximately continuous integrals and wide Denjoy integral had been for a long time a matter of interest. Wide Denjoy integration is less restrictive in the sense of variation of indefinite integrals, and so, it turns out, there are Denjoy integrable functions that are not Burkill integrable (and even approximately Kurzweil–Henstock integrable). First example of such a function has been given by Tolstov [13]; see also [2]. Since wide Denjoy and Burkill integrals are noncomparable (as mentioned above, exact approximate derivatives with discontinuous primitives are not Denjoy integrable), it was natural to ask for a common extension for both of them. There were various integration processes proposed, see for instance [3, 4, 5, 9, 11], all being straightforward extensions of Lusin's definition of wide Denjoy integral via generalization of absolute continuity ($ACG$-function) and most of, at the time, lacking a correct proof of either extending also Burkill's (or the approximately continuous Kurzweil–Henstock) integral,

[1] Here, *exact* approximate derivative means this derivative exists everywhere; i.e., the given function is everywhere approximately differentiable.

or being uniquely defined. Only later on correct proofs have been provided [6, 7, 10]. It was shown that the smallest of the extensions proposed (Kubota integral) indeed covers approximate Kurzweil–Henstock integral. See [12] for a brief history of the problem and a relation chart for the solutions.

We say an $F\colon \langle a,b\rangle \to \mathbb{R}$ is $[ACG]$ if $\langle a,b\rangle = \bigcup_{n=1}^{\infty} E_n$, where for each $n$, $E_n$ is closed and the restriction $F \restriction E_n$ is absolutely continuous.

**Definition 1.** *We say that an $f\colon \langle a,b\rangle \to \mathbb{R}$ is Kubota integrable, if there exists an approximately continuous $[ACG]$-function $F\colon \langle a,b\rangle \to \mathbb{R}$ such that $F'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in \langle a,b\rangle$. The integral of $f$ is defined as $F(b) - F(a)$.*

In this talk the problem if Kubota integral is the smallest extension possible for its purpose will be discussed in a wider context of approximately continuous and Baire one star functions.

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