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WHERE DOES TAKAGI'S CONTINUOUS, NOWHERE DIFFERENTIABLE FUNCTION HAVE AN INFINITE DERIVATIVE?

1 Introduction.

Takagi's function [5] is defined by

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \phi^{(n)}(x), \qquad 0 \le x \le 1,$$

where $\phi^{(1)} := \phi$ is the "tent map" defined by

$$\phi(x) := \begin{cases} 2x, & \text{if } 0 \le x \le 1/2, \\ 2 - 2x, & \text{if } 1/2 \le x \le 1; \end{cases}$$

and inductively, $\phi^{(n)} := \phi \circ \phi^{(n-1)}$ for $n \ge 2$. The function T is continuous and does not have a finite derivative anywhere. But at which points does it have an *infinite* derivative? This question appeared to be settled in 1936 by Begle and Ayres [2]. Let O_n be the number of zeros, and $I_n = n - O_n$ the number of ones, among the first n binary digits of x, and let $D_n = O_n - I_n$. Begle and Ayres

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claimed that $T'(x) = \infty$ if $D_n \to \infty$, and $T'(x) = -\infty$ if $D_n \to -\infty$. But their proof contained several mistakes. In fact, Krüppel [4] recently published the following counterexample to their claim: Let $x = \sum_{n=1}^{\infty} 2^{-a_n}$, where $a_n = 4^n$. For this x, we certainly have $D_n \to \infty$. But one can show that the secant slopes over a suitably chosen sequence of dyadic intervals about x tend in fact to $-\infty$, so that a derivative of $+\infty$ is impossible. (See [1, Section 2].)

The purpose of this article is to give a complete characterization of the points x at which T has an infinite derivative. Since the condition we obtain is somewhat opaque, we illustrate it with several examples. This is done in Section 2. The proofs appear in [1]. In Section 3 we extend a recent result of Krüppel [4] concerning the modulus of continuity of T.

2 Improper derivatives.

It is well known (e.g. [2, 4]) that if x is a dyadic rational, then the right and left derivatives of T at x are $+\infty$ and $-\infty$, respectively. We now treat the non-dyadic case.

Theorem 1. *Let $x \in (0,1)$ be non-dyadic, and write*

$$x = \sum_{n=1}^{\infty} 2^{-a_n}, \qquad 1 - x = \sum_{n=1}^{\infty} 2^{-b_n}, \tag{1}$$

where $\{a_n\}$ and $\{b_n\}$ are strictly increasing sequences of positive integers, determined uniquely by x. Then:

(i) $T'(x) = +\infty$ if and only if

$$a_{n+1} - 2a_n + 2n - \log_2(a_{n+1} - a_n) \to -\infty. \tag{2}$$

(ii) $T'(x) = -\infty$ if and only if

$$b_{n+1} - 2b_n + 2n - \log_2(b_{n+1} - b_n) \to -\infty. \tag{3}$$

In fact, (ii) follows directly from (i) by the symmetry of the Takagi function: $T(x) = T(1-x)$ for $0 \le x \le 1$. So it is sufficient to prove (i).

Remark 1. Conditions (2) and (3) may look a bit mysterious. The examples below aim to shed more light on them. Since the conditions are quite analogous, we focus on (2).

Example 1. If $D_n \to \infty$ and the number of consecutive 0's in the binary expansion of x is bounded, then $T'(x) = +\infty$. Similarly, if $D_n \to -\infty$ and the number of consecutive 1's is bounded, then $T'(x) = -\infty$. $\boxed{\star}$

Example 2. If $\limsup_{n\to\infty} a_{n+1}/a_n > 2$, then (2) fails. $\boxed{\star}$

Example 3. A sufficient condition for (2) to hold is that, for some $0 < \varepsilon \leq 1$,

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2 - \varepsilon \qquad \text{and} \qquad \liminf_{n\to\infty} \frac{a_n}{n} > \frac{2}{\varepsilon}. \tag{4}$$

For instance, (2) holds for $a_n = 3n$; for any increasing polynomial of degree 2 or higher; and for any exponential sequence $a_n = \lfloor \alpha^n \rfloor$ with $1 < \alpha < 2$. It also holds when a_n is the n-th prime number. $\boxed{\star}$

Example 4. The sequence $a_n = 2^n$ does not satisfy (2); neither does $a_n = 2^n + n$. But $a_n = 2^n + (1+\varepsilon)n$ satisfies (2) for any $\varepsilon > 0$. In this example, the logarithmic term in (2) makes all the difference. $\boxed{\star}$

An important subset of $[0,1]$ is formed by the points x whose binary expansion has a density; that is, points $x = \sum_{k=1}^{\infty} 2^{-k}\varepsilon_k$ for which the limit

$$d_1(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \tag{5}$$

exists. Note that $d_1(x)$ expresses the long-run proportion of 1's in the binary expansion of x. If it exists, we define

$$d_0(x) := 1 - d_1(x) \tag{6}$$

to denote the long-run proportion of 0's.

Corollary 1. *Let x be a non-dyadic point, and suppose $d_1(x)$ exists.*

(i) If $0 < d_1(x) < 1/2$, then $T'(x) = +\infty$.

(ii) If $1/2 < d_1(x) < 1$, then $T'(x) = -\infty$.

(iii) If $d_1(x) = 0$ and $\limsup_{n\to\infty} a_{n+1}/a_n < 2$, then $T'(x) = +\infty$.

(iv) If $d_1(x) = 1$ and $\limsup_{n\to\infty} b_{n+1}/b_n < 2$, then $T'(x) = -\infty$.

Using Corollary 1 and a result of Besicovitch, we get

Corollary 2. *Let $\dim_H$ denote Hausdorff dimension. Then we have*

$$\dim_H\{x : T'(x) = \infty\} = \dim_H\{x : T'(x) = -\infty\} = 1.$$

Corollary 1 left out the binary *normal numbers*; that is, those numbers x for which $d_1(x) = 1/2$. For a discussion of this case, we refer to the full paper [1].

3 The modulus of continuity.

In this final section we present an exact result concerning the modulus of continuity of T. Let $d_1(x)$ and $d_0(x)$ denote the densities of 1 and 0 in the binary expansion of x, respectively, as in (5) and (6).

Definition 1. A point $x \in [0,1]$ is *density-regular* if $d_1(x)$ exists and one of the following holds:
(a) $0 < d_1(x) < 1$; or
(b) $d_1(x) = 0$ and $a_{n+1}/a_n \to 1$; or
(c) $d_1(x) = 1$ and $b_{n+1}/b_n \to 1$.

Here, $\{a_n\}$ and $\{b_n\}$ are the sequences determined by (1).

The following theorem extends a result of Krüppel [4], who proved it for rational points x with a nonterminating binary expansion.

Theorem 2. *For non-dyadic x, the limit*

$$\lim_{h\to 0} \frac{T(x+h) - T(x)}{h \log_2(1/|h|)}$$

exists if and only if x is density-regular, in which case the limit is equal to $d_0(x) - d_1(x)$.

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