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THE DERIVATIVE OF LEBESGUE'S SINGULAR FUNCTION

Imagine flipping an unfair coin with probability $a \in (0,1)$ of heads and probability $1-a$ of tails. Note that $a \neq 1/2$. Let the binary expansion of $t \in [0,1]$: $t = \sum_{n=1}^{\infty} \omega_n/2^n$ be determined by flipping the coin infinitely many times. More precisely, $\omega_n = 0$ if the n-th toss is heads and $\omega_n = 1$ if it is tails. *Lebesgue's singular function* $L_a(x)$ is defined as the distribution function of t:

$$L_a(x) := prob\{t \leq x\}, \qquad 0 \leq x \leq 1.$$

This function was also defined in different ways and studied by a number of authors: Cesaro (1906), Faber (1910), Lomnicki and Ulam(1934), Salem (1943), De Rham (1957) and others, and therefore $L_a(x)$ is also called Salem's singular function or De Rham's singular function. In recent years, several applications have been reported: for instance, in physics [11], [12], number theory [8], [6] and dynamical systems [2], [9].

It is well-known that

1. $L_a(x)$ is strictly increasing, but the derivative is 0 almost everywhere.
2. For any value of $x \in [0,1]$, the derivative is either zero, or $+\infty$, or it does not exist.

It is natural to ask at which points $x \in [0,1]$ exactly we have $L_\alpha^{'}(x) = 0$ or $+\infty$. In fact, De Rham [3] gave the following answer : Let the binary expansion of $x \in [0,1]$ be $x = \sum_{k=1}^{\infty} 2^{-k}\varepsilon_k$, where $\varepsilon_k \in \{0,1\}$. Note that for those $x \in [0,1]$ having two binary expansions, we choose the expansion which is eventually all zeros. As an exception, fix $\varepsilon_k = 1$ for every k if $x = 1$. Define

$$I_n := \sum_{k=1}^{n} \varepsilon_k.$$

Note that I_n is the number of 1's occurring in the first n binary digits of x. Suppose that I_n/n tends to a limit l as $n \to \infty$, and let

$$l_0 := \frac{\log 2a}{\log a - \log(1-a)}.$$

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Then the derivative of $L_a(x)$ exists and is zero, when $(l - l_0)(a - 1/2) > 0$. An English translation of De Rham's original paper is included in Edgar's book [4]. Unfortunately, De Rham's paper did not contain a proof. The main purpose of this note is to give a proof of De Rham's statement and extend his result.

Define

$$D_1(x) := \lim_{n\to\infty} \frac{I_n}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \varepsilon_k, \tag{1}$$

provided the limit exists, and put $D_0(x) := 1 - D_1(x)$. In other words, $D_i(x)$ is the density of the digit i in the binary expansion of x, for $i = 0, 1$.

Theorem 1. *1. If $x \in [0,1]$ is dyadic, then $L'_{a+}(x) \neq L'_{a-}(x)$.*

2. If $x \in [0,1]$ is not dyadic and $0 < D_1(x) < 1$, then

$$L'_a(x) = \begin{cases} 0, & \textit{if} \quad a^{D_0(x)}(1-a)^{D_1(x)} < 1/2, \\ +\infty, & \textit{if} \quad a^{D_0(x)}(1-a)^{D_1(x)} > 1/2. \end{cases}$$

What kind of applications does Theorem 1 have? In fact, an application arises from the following simple question. In classical calculus, the chain rule is used to compute the derivative of the composition of two differentiable functions. However, what can we say about the differentiability of the composition of a nowhere differentiable function f and a singular function g ? For instance, if f is well-known *Takagi's nowhere differentiable function* which is defined by

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k}|2^k x - \lfloor 2^k x + \tfrac{1}{2}\rfloor|, \qquad 0 \le x \le 1,$$

and g is the the inverse of Lebesgue's singular function, then is $(T \circ L_a^{-1})(x)$ nowhere differentiable?

Although T does not have a finite derivative anywhere, it is known to have an improper infinite derivative at many points. In fact, Allaart and Kawamura recently proved that the set of points where $T'(x) = +\infty$ or $-\infty$ has Hausdorff dimension one [1]. Note that the inverse of Lebesgue's singular function is also singular. Hence, if we try to (naively) use the chain rule to compute the derivative of $(T \circ L_a^{-1})(x)$, we may run into one of the indeterminate products $+\infty \cdot 0$, or $-\infty \cdot 0$.

The following theorem gives an answer to this concrete question: $(T \circ L_a^{-1})(x)$ has a finite derivative at uncountably many points.

Theorem 2. *Let $x \in [0,1]$, and put $y = L_a^{-1}(x)$. If $0 < D_1(y) < 1$ and $a^{D_0(y)}(1-a)^{D_1(y)} > 1/2$, then*

$$(T \circ L_a^{-1})'(x) = 0.$$

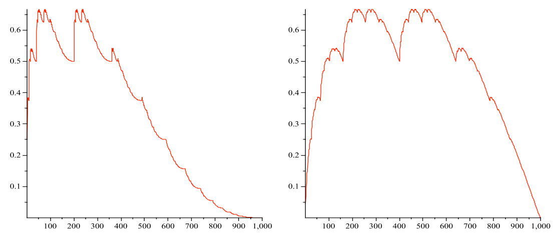

Figure 1: Graphs of $(T \circ L_a^{-1})(x)$ for $a = 0.2$ and $a = 0.4$

References

[1] P. Allaart and K. Kawamura, The improper infinite derivatives of Takagi's nowhere-differentiable function, *J. Math. Anal. Appl.*, **372** (2010), 656–665.

[2] L. Berg and M. Kruppel, De Rham's singular function and related functions, *Z. Anal. Anwendungen.*, **19(1)** (2000), 227–237.

[3] G. de Rham, Sur quelques courbes définies par des équations fonctionnelles, *Rend. Sem.Mat.Torino* **16** (1957), 101–113.

[4] G. A. Edgar, *Classics on Fractals*, Addison-Wesley, Reading, MA 1993.

[5] K. Kawamura, At which points exactly has Lebesgue's singular function the derivative zero?, Preprint, 2010.

[6] M. Kruppel, De Rham's singular function, its partial derivatives with respect to the parameter and binary digital sums, *Rostock. Math. Kolloq.*, **64**, (2009) 57–74.

[7] Z. Lomnicki and S. Ulam, Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités I. Variables indépendantes. *Fund. Math.* **23** (1934), 237–278.

[8] T. Okada, T. Sekiguchi and Y. Shiota, An explicit formula of the exponential sums of digital sums, *Japan J. Indust. Appl. Math.* **12** (1995), 425–438.

[9] H. Sumi, Rational semigroups, random complex dynamics and singular functions on the complex plane. *Sugaku* **61(2)** (2009), 133–161.

[10] T. Takagi, A simple example of the continuous function without derivative, *Phys.-Math. Soc. Japan* **1** (1903), 176–177.

[11] H. Takayasu, Physical models of fractal functions, *Japan J. Appl. Math.*, **1** (1984), 201–205.

[12] S. Tasaki, I. Antoniou and Z. Suchanecki, Deterministic diffusion, De Rham equation and fractal eigenvectors, *Physics Letter* A **179 (1)** (1993) 97–102.