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A NATURAL EXTENSION OF THE HENSTOCK-KURZWEIL INTEGRAL

1 Introduction

In 1990, Ralph Henstock and Jaroslav Kurzweil proposed a modification to the definition of the classical Riemann integral which is equivalent to the Denjoy -Perron integral. This modified definition gave birth to a definition of the integral that is known as the "gauge integral". The book of R. G. Bartle [1] presents in considerable detail an introduction to this Henstock-Kurzweil integral. The techniques used in the gauge integral are tied up with the metric properties of the Euclidean space $\mathbb{R}^k$. Such an approach is not suitable in a broader sense when generalization of the integral on abstract spaces is concerned. The purpose of this talk is to propose a modification towards a natural extension of the Henstock-Kurzweil integral on abstract spaces. This natural modification will help in proving many of the basic and fundamental theorems of integration on abstract spaces.

2 Preliminaries

Throughout this paper, a measurable space is a pair (Ω, Σ) where Ω is a topological space, Σ is a $\sigma-$algebra containing the Borel $\sigma-$algebra $\mathcal{B}_\Omega$ of Ω. The elements of the $\sigma-$algebra Σ are referred as measurable sets. In what follows, we will use $\bigsqcup_{n=1}^{\infty} A_n$ to indicate the union of the disjoint family of sets (A_n).

By a measure we mean a set function $\eta : \Sigma \to \mathbb{R} \cup \{\pm\infty\}$ that satisfies $\eta(\emptyset) = 0$, and η is countably additive; that is, $\eta(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty}(A_n)$ whenever (A_n) is a disjoint family in Σ. Note that we do not require a measure η to be nonnegative. A measure η can take either only $+\infty$ or $-\infty$.

Given a nonnegative measure μ and a measurable set E, let

$$\kappa_\mu(E) = \{K \in \Sigma : K \text{ is compact}, K \subset E, \mu(K) < \infty\}.$$

We propose the following definition.

Definition 1. A nonnegative measure μ on the measurable space Ω is said to have the compact measure property (CMP) if for every $E \in \Omega$,

$$\mu(E) = \sup(\mu(K) : K \in \kappa_\mu(E)).$$

3 σ-Simple Function

We generalize the notion of simple function as follows:

Definition 2. A function $\phi : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is said to be a σ-simple function if it can be expressed as a series of the form $\phi = \sum_{n\in N} c_n I_{A_n}$ where $A_n|_{n=1}^{\infty}$ is a disjoint family of elements of Σ, I_{A_i} is the indicator function of the set A_i and (c_n) is a sequence in $\mathbb{R}$ such that $\sum_{n\in N} |c_n| < \infty$. We will denote by $S(\Omega, \Sigma)$ the class of all σ - simple functions. We now define integrability of a simple function.

Definition 3. We say that the σ-simple function $\phi = \sum_{n\in N} c_n I_{A_n}$ is integrable over a measurable set E provided that the series $\sum_{n\in} |c_n|\mu(A_n \bigcap E)$ converges. The integral of ϕ over E is defined by

$$\int_E \phi = \sum_{n\in N} c_n \mu(A_n \bigcap E).$$

We denote by $S(E, \Sigma, \mu)$ the set of all integrable σ-simple function over the measurable set E.

4 Integrability and Integral

Definition 4. A function $f : \Omega \to R$ is said to be μ- integrable over a Σ- measurable set E if there exists a nondecreasing sequence (ϕ_n) and a nonincreasing sequence (ψ_n) of σ-simple functions in $S(E, \Sigma, \mu)$ such that

1. $\phi_n \leq f \leq \psi_{n)}$ almost everywhere on E;
2. $\int_K (\psi_n - \phi_n) \to 0$ for every K compact subset of E.

We denote by $\tau(E, \Sigma, \mu)$ the space of all μ- integrable functions on the measurable set E. It is clear that $S(E, \Sigma, \mu) \subset \tau(E, \Sigma, \mu)$.

The pair (ϕ_n, ψ_n) in the above definition is called a generating pair for f over E.

Definition 5. Let $f : \Omega \to R$ be a μ-integrable over a Σ- measurable set E. We define the integral of f to be

$$\int_E f = \lim_{K \in \kappa_\mu(E), K \uparrow E} \int_K f.$$

5 Main Result

Theorem 1 (Monotone Convergence Theorem)**.** *Let (E, Σ, μ) be a measure space and let $E \in \Sigma$. Let (f_n) be a monotone sequence in $\tau(E, \Sigma, \mu)$ and let $f = \mathrm{esslim} f_n$. Then $f \in \tau(E, \Sigma, \mu)$ if and only if $\sup(\int_E f_n : n \in N) < \bar{}\infty$. In this case, $\int_E f = \lim \int_E f_n$.*

Theorem 2 (Fatou's Lemma)**.** *Let $(f_n) \subset \tau(E, \Sigma, \mu)$ where $E \in \Sigma$. Suppose that*

(i) *there exists $\alpha \in \tau(E, \Sigma, \mu)$ such that $\alpha(x) \leq f_n(x)$, for almost every $x \in E, n \in N$,*

(ii) $\liminf \int_E f_n < \infty$.

Then

(a) $\liminf f_n \in \tau(E, \Sigma, \mu)$ *and*

(b) $-\infty < \int_E \liminf f_n \leq \liminf \int_E f_n < \infty$.

Theorem 3 (Dominated Convergence Theorem)**.** *Let $(f_n) \subset \tau(E, \Sigma, \mu)$ where $E \in \Sigma$. Suppose that*

(i) *(f_n) converges almost everywhere to f,*

(ii) *there exist $\alpha, \beta \in \tau(E, \Sigma, \mu)$ such that $\alpha(x) \leq f_n(x) \leq \beta(x)$, for almost every $x \in E$, for every $n \in N$.*

Then

(a) $f = \mathrm{esslim} f_n \in \tau(E, \Sigma, \mu)$ *and* $\int_E f = \lim \int_E f_n < \infty$*;*

(b) $|f - f_n| \in \tau(E, \Sigma, \mu)$ *and* $\lim \int_E |f - f_n| = 0$.

6 Comment

This presentation is the outcome from my joint work with M.A. Robdera.

References

[1] R. G. Bartle, *A Modern Theory of Integration*, Graduate Studies in Mathematics, AMS, 2001.

[2] H. S. Bear, *A Primer of Lebesgue Integration*, 2nd Edition, Academic Press, 2002.

[3] I. K. Rana, *An Introduction to Measure and Integration*, Graduate Studies in Mathematics, AMS, 2nd Edition, Narosa Publishing House,2002.