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LINEAR FDES IN THE FRAME OF GENERALIZED ODES: A JUSTIFICATION FOR USING KURZWEIL EQUATIONS

This presentation is based on results contained in a joint work (in preparation) with professor Štefan Schwabik.

In [5] and [4], it was proved that retarded functional differential equations (we write FDEs for short) and impulsive FDEs can be related to generalized ODEs (also known as Kurzweil equations) and several nice applications coming from this relation appear. See, for instance, [1], [2] and [3].

Our aim here is to present a simple relation between linear FDEs of type

$$\begin{cases} \dot{y} = \mathcal{L}(y_t, t), \\ y_{t_0} = \varphi, \end{cases} \tag{1}$$

where $\mathcal{L}$ is a linear bounded operator and φ is a continuous function, and linear generalized ODEs of the form

$$\frac{dx}{d\tau} = D[A(t)x], \qquad x(t_0) = \widetilde{x}. \tag{2}$$

This relation is very interesting and it leads us to important applications.

Let $J \subset \mathbb{R}$ be an interval and $L(X)$ be the Banach space of bounded linear operators from X into itself endowed with the usual operator norm in $L(X)$. Given $(\widetilde{x}, t_0) \in X \times J$, let $\Omega = \overline{B}_c(\widetilde{x}) \times [a, b]$, where $B_c(\widetilde{x}) = \{x \in X;\ \|x - \widetilde{x}\| \leq c\}$, $c > 0$, and $[a, b] \subset J$.

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Let us assume that $F(x,t) = A(t)x$, where $A : J \to L(X)$ is left continuous and locally of bounded variation, that is, for each $[a,b] \subset J$, $\mathrm{var}_a^b(A) < +\infty$. Moreover, we assume that the operator $I + [A(t+) - A(t)] = I + \Delta^+ A(t)$ is invertible for every $t \in [a,b]$ ($I \in L(X)$ is the identity operator and $A(t+) = \lim_{s \to t+} A(s)$).

Consider the initial value problem for a linear generalized ODE in the form (2). The function $x : [a,b] \to X$ is a solution of (2) on $[a,b]$ if

$$x(s) = \widetilde{x} + \int_{t_0}^{s} D[A(t)x(\tau)], \tag{3}$$

for all $s \in [a,b]$, where the integral is in the sense of Jaroslav Kurzweil (see [6] and [7]).

By the properties of the Kurzweil integral, the integral on the righthand side of (3) is formed by Stieltjes-type integral sums. Therefore (3) can be rewritten as

$$x(s) = \widetilde{x} + \int_{t_0}^{s} d[A(r)]x(r), \qquad s \in [a,b]. \tag{4}$$

Denote by $C([a,b],\mathbb{R}^n)$ the Banach space, endowed with the usual supremum norm, of continuous functions from the compact interval $[a,b] \subset \mathbb{R}$ to $\mathbb{R}^n$. Let $r, \sigma > 0$ and $t_0 \in \mathbb{R}$. Given a function $y : \mathbb{R} \to \mathbb{R}^n$, let $y_t : [-r,0] \to \mathbb{R}^n$ be given by

$$y_t(\theta) = y(t+\theta), \qquad \theta \in [-r,0], \quad t \in \mathbb{R}.$$

It is clear that for $t_0 \in \mathbb{R}$ and a function $y \in C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, we have $y_t \in C([-r,0],\mathbb{R}^n)$ for all $t \in [t_0, t_0 + \sigma]$.

Let $G_1 \subset C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ be an open set with the following property: if y is an element of G_1 and $\bar{t} \in [t_0, t_0 + \sigma]$, then $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & t_0 - r \le t \le \bar{t} \\ y(\bar{t}), & \bar{t} < t \le t_0 + \sigma \end{cases}$$

is also an element of G_1. In particular, any open ball in $C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ has this property.

Consider the initial value problem for a linear retarded FDE in the form (1), where $\phi \in C([-r,0],\mathbb{R}^n)$ and $\mathcal{L} : C([-r,0],\mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is bounded and linear. Then there exists $\eta : \mathbb{R} \times \mathbb{R} \to L(\mathbb{R}^n)$, with $\eta(t,\theta) = 0$ for $\theta \ge 0$, and $\eta(t,\theta) = \eta(t,-r)$ for $\theta \le -r$, for which $\eta(t,\cdot)$ is left continuous and of bounded variation on $[-r,0]$ for a fixed t and such that

$$\mathcal{L}(\psi, t) = \int_{-r}^{0} d_\theta[\eta(t,\theta)]\psi(\theta).$$

Define

$$m(t) = \operatorname{var}_{-r}^{0} \eta(t, \cdot), \quad t \in [t_0, t_0 + \sigma],$$

and assume $m \in L_1([t_0, t_0 + \sigma])$. Then for $t \in [t_0, t_0 + \sigma]$ and $\psi \in C = C([-r, 0], \mathbb{R}^n)$,

$$|\mathcal{L}(\psi, t)| = \left| \int_{-r}^{0} d_\theta[\eta(t, \theta)]\psi(\theta) \right| \leq \operatorname{var}_{-r}^{0} \eta(t, \cdot)\|\psi\|_C = m(t)\|\psi\|_C.$$

Thus, for $y \in C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, we have

$$|\mathcal{L}(y_t, t)| \leq m(t) \cdot \|y_t\|_C \leq m(t) \cdot \|y\|_{C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)}$$

and hence $t \mapsto \mathcal{L}(y_t, t)$ belongs to $L_1([t_0, t_0 + \sigma], \mathbb{R}^n)$.

For $y \in C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ and $t \in [t_0, t_0 + \sigma]$, define

$$L(y, t)(\vartheta) = \begin{cases} 0, \quad t_0 - r \leq \vartheta \leq t_0, \\ \int_{t_0}^{\vartheta} \int_{-r}^{0} d_\theta[\eta(s, \theta)]y(s + \theta)ds, \quad t_0 \leq \vartheta \leq t \leq t_0 + \sigma, \\ \int_{t_0}^{t} \int_{-r}^{0} d_\theta[\eta(s, \theta)]y(s + \theta)ds, \quad t_0 \leq t \leq \vartheta \leq t_0 + \sigma. \end{cases}$$

Then $L(\cdot, t) : C([t_0 - r, t_0 + \sigma], \mathbb{R}^n) \to C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$. Also, $L(y, t)$ is linear on the first variable. Moreover, for $L(y, t) = A(t)y$, we have $A(t) : C([t_0 - r, t_0 + \sigma], \mathbb{R}^n) \to C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, for $t \in [t_0 - r, t_0 + \sigma]$. Thus, considering the norm $\|A(t)y\|_{C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)} = \sup_{\vartheta \in [t_0 - r, t_0 + \sigma]} |L(y, t)(\vartheta)|$, we have

$$\|A(t)y\|_{C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)} \leq \int_{t_0}^{t} m(s)ds \cdot \|y\|_{C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)}.$$

Hence the linear operator $A(t)$ is bounded, $A : [t_0 - r, t_0 + \sigma] \to L(C([t_0 - r, t_0 + \sigma], \mathbb{R}^n))$ and

$$\|A(t)\|_{L(C([t_0 - r, t_0 + \sigma], \mathbb{R}^n))} \leq \int_{t_0}^{t} m(s)ds.$$

Now we have the following two results connecting problem (1) with problem (2). Their proofs can be found in [4], Theorems 3.4 and 3.5.

Theorem 1. *Let y be a solution of problem* (1) *on* $[t_0, t_0 + \sigma]$. *Given* $t \in [t_0 - r, t_0 + \sigma]$, *put*

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), \ \vartheta \in [t_0 - r, t] \\ y(t), \ \vartheta \in [t, t_0 + \sigma]. \end{cases}$$

Then $x(t) \in C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *and x is a solution of problem* (2) *on* $[t_0 - r, t_0 + \sigma]$.

Theorem 2. *Let x be a solution of the problem* (2) *on* $[t_0 - r, t_0 + \sigma]$. *For every* $\vartheta \in [t_0 - r, t_0 + \sigma]$, *let*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \leq \vartheta \leq t_0 \\ x(\vartheta)(\vartheta), & t_0 \leq \vartheta \leq t_0 + \sigma. \end{cases}$$

Then y is a solution of the problem (1) *on* $[t_0 - r, t_0 + \sigma]$.

This two results produce a one-to-one correspondence between solutions of (1) and (2) and open the way of translating results from (2) to problem (1).

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