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Paul Musial, Chicago State University, Department of Mathematics and Computer Science, 9501 South King Drive Chicago, IL 60628. email: pmusial@csu.edu

THE L^r-VARIATIONAL INTEGRAL

We define the class of L^r-variational integrable functions and show that it is equivalent to the class of L^r- Henstock Kurzweil integrable functions. We also define the class of functions of L^r-bounded variation.

In [3] we have the following definition: let $r \geq 1$ and let $f : [a,b] \to R$. We say that f is L^r-Henstock-Kurzweil integrable on $[a,b]$ ($f \in HK_r([a,b])$) if there exists a function $F \in L^r([a,b])$ so that for any $\varepsilon > 0$ there exists a gauge $\delta > 0$ defined on $[a,b]$ so that if $P = \{(x_i,[c_i,d_i])\}_{i=1}^n$ is a δ-fine tagged partition of $[a,b]$ then

$$\sum_{i=1}^n \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} < \varepsilon.$$

Let $r \geq 1$ and let $f : [a,b] \to R$. We say that f is *L^r-variational integrable* on $[a,b]$ if there exists a function $F \in L^r([a,b])$ having the following property: for any $\varepsilon > 0$ there exists a non-decreasing function $\phi : [a,b] \to R$ and a gauge $\delta > 0$ so that $\phi(b) - \phi(a) < \varepsilon$ and whenever $(x,[c,d])$ is a δ-fine tagged subinterval of $[a,b]$ we have

$$\left(\frac{1}{d-c} \int_c^d |F(y) - F(x) - f(x)(y-x)|^r \, dy \right)^{1/r} < \phi(d) - \phi(c).$$

Theorem 1: $f \in HK_r([a,b])$ if and only if f is L^r-variational integrable on $[a,b]$.

Sketch of proof: We proceed in a manner similar to that in which the class of Henstock-Kurzweil integrable functions is shown to be equivalent to the class of variational integrable functions. See [2].

We now define the class of functions of L^r-bounded variation. Let $r \geq 1$, let $f : [a,b] \to R$ and let E be a measurable subset of $[a,b]$. We say that

f is *L^r-bounded variation on* E ($f \in BV_r([E])$) if there exist $M > 0$ and a gauge $\delta > 0$ defined on E so that if $P = \{(x_i, [c_i, d_i])\}_{i=1}^n$ is a finite collection of δ-fine tagged subintervals of $[a, b]$ having tags in E, then

$$\sum_{i=1}^{n} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r \, dy \right)^{1/r} < M.$$

Theorem 2: If $f \in BV_r(E)$ then we can find $\{E_n\}_{n \geq 1}$ so that

$$E = \bigcup_{n=1}^{\infty} E_n$$

and $f \in BV(E_n)$ [2] for all n.

Sketch of proof: We proceed in a manner similar to that in which it is shown that if F is absolutely continuous in L^r sense on E then there exist $\{E_n\}$ so that $E = \cup_{n=1}^{\infty} E_n$ and F is absolutely continuous on E_n for all n. See [3].

References

[1] L. Gordon, *Perron's Integral for Derivatives in L^r*, Studia Mathematica, **28** (1967), 295–316.

[2] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, Graduate Studies in Mathematics, 4, American Mathematical Society, 1994.

[3] P. Musial and Y. Sagher, *The L^r Henstock-Kurzweil Integral*, Studia Mathematica, **160** (2004), 53–81.