Real Analysis Exchange
Summer Symposium 2010, pp. 103–105

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BASIS PROPERTIES OF THE EIGENSYSTEM OF A PERTURBED HARMONIC OSCILLATOR

Abstract

We consider the perturbed harmonic oscillator $L = T + B$, with $T = -d^2/dx^2 + x^2$ and $B = b(x)$ with domain in $L^2(\mathbb{R})$. In [1] it is shown that the eigensystem of L forms an unconditional basis for $L^2(\mathbb{R})$ when b belongs to certain function spaces, for example $L^p(\mathbb{R})$ when $2 \leq p < \infty$. These results are summarized.

1 Introduction.

Consider the harmonic oscillator

$$T = -\frac{d^2}{dx^2} + x^2.$$

The normalized eigenfunctions of T form an orthonormal basis in $L^2(\mathbb{R})$. They are the well-known Hermite functions $h_k(x)$, $\|h_k\|_2 = 1$ (see for example [6, Ch. 5, Sect. 4] , [7, Ch XII, Sect. 6.4]). The eigenvalue associated with the eigenfunction h_k is $\lambda_k = 2k+1$, i.e. $Th_k = (2k+1)h_k$. Consider a perturbation of T, $L = T + B$ where B satisfies $\mathrm{dom}T \subset \mathrm{dom}B$. If

$$\|B\| < 1 \tag{1}$$

then it can be shown that the eigensystem of L forms an unconditional basis for $L^2(\mathbb{R})$. Furthermore, it is possible to construct an operator B with $\|B\| = 1$ such that the eigensystem of L is not a basis at all (see [1, section 6]), so

Mathematical Reviews subject classification: Primary: 47E05, 34L40; Secondary: 34L10
Key words: Harmonic oscillator, Hermite functions, discrete Hilbert transform, unconditional basis
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condition (1) cannot generally be improved. In [1], we consider the case where B is a multiplication operator

$$Bf = b(x)f(x).$$

In this case, condition (1) can be greatly improved (see Theorem 1).

2 Main Results.

Consider the following conditions on b:

$$b \in L^p(\mathbb{R}) \quad \text{for some } p \text{ with} \quad 2 \le p < \infty, \tag{2a}$$

$$b \in L_0^\infty(\mathbb{R}) = \{\phi \in L^\infty(\mathbb{R}) : \lim_{T\to\infty} \text{ess sup}_{|t|>T}|\phi(t)| = 0\}, \tag{2b}$$

$$b \in L_\zeta(\mathbb{R}) = \{\phi : \phi(x)/(1+|x|)^\zeta \in L^2(\mathbb{R})\} \quad \text{with} \quad \zeta < 1/6. \tag{2c}$$

The main result of [1] is the following.

Theorem 1. *Let b belong to any one of the spaces (2a)-(2c). Then the spectrum of L is discrete and eventually simple. The eigensystem for L forms an unconditional basis in $L^2(\mathbb{R})$.*

The proof of this theorem uses the following lemma of Kato which gives sufficient conditions for an eigensystem of a perturbation of a self-adjoint operator to be an unconditional basis [4], [5, Ch5, Lemma 4.17a].

Lemma 2. *Let $\{Q_k^0\}_{j\in\mathbb{Z}_+}$ be a complete family of orthogonal projections in a Hilbert space X and let $\{Q_k\}_{j\in\mathbb{Z}_+}$ be a family of (not necessarily orthogonal) projections such that $Q_jQ_k = \delta_{j,k}Q_j$. Assume that*

$$dim(Q_0^0) = dim(Q_0) = m < \infty \tag{3}$$

$$\sum_{j=1}^{\infty} \|Q_j^0(Q_j - Q_j^0)u\|^2 \le c_0\|u\|^2, \quad \textit{for every} \quad u \in X \tag{4}$$

where c_0 is a constant smaller than 1. Then there is a bounded operator $W : X \to X$ with bounded inverse such that $Q_j = W^{-1}Q_j^0W$ for $j \in \mathbb{Z}_+$.

Other ingredients of the proof of Theorem 1 in [1] include asymptotic estimates for the Hermite functions [8, Lemma 1.5.1, Lemma 1.5.2, p.26-27], [2, Lemma 4] and boundedness of the discrete Hilbert transform considered on certain weighted $\ell^2(\mathbb{N})$ spaces [3, Sect. 8.12 statement 294], [1, Appendix].

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