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THE GOLDEN ANA SET

The Golden Ana Set was first defined by Clifford Pickover[2], and was further studied (in various interpretations) by Joseph Pe [1]. It's construction is based on an infinite sequence of strings on the set of letters $\{a,\ n\}$ defined by a replacement rule. We start with an initial a, and every term is constructed by replacing a's in the preceding term by ana, and n's by an.

a
ana
ana an ana
ana an ana ana an ana an ana
ana an ana ana an ana an ana ana an ana ana an ana an ana ana an ana an ana

We will say $G_1 = a$, and G_{i+1} is obtained from G_i by using the replacement rule. For each G_i in our sequence, we have a geometric interpretation for our string of letters as subsets of the unit interval. If our string G_i is length k, we partition the unit interval into k subintervals of equal length $\frac{1}{k}$. We label the subintervals I_j, where the $1 \leq j \leq k$, and If $t < j$, then $x \in I_t$ and $y \in I_j \Rightarrow x < y$, i.e. the intervals are indexed with values increasing from left to right. If the j^{th} letter in G_i is a, then I_j is closed, and if the j^{th} letter is n, then I_j is open. We define $A_i = \bigcup I_j$ such that the j^{th} letter in G_i is a. We now have a sequence of subsets of the unit interval A_n.

In this paper, we define the Golden Ana Set $G = \bigcap_{i=1}^{\infty} A_i$. To clarify, when we say G_i, we are talking about the i^{th} string of letters in G_n, and when we say A_i, we mean the corresponding subset of $[0, 1]$ defined above. The primary focus of this paper is to determine what points actually lie in G. To determine if a point $x \in [0, 1]$ is also in G, we must show that it is in A_i for every $i \in \mathbb{N}$. To do this, let I_{i_k} denotes the k^{th} interval in A_i. We introduce a techniques to determine if an interval I_{i_k} that contains x corresponds to a or n in G_i for each $i \in\ \mathbb{N}$. Let M_i denote the number of letters in the string G_i. Since our intervals in A_i are indexed from left to right by their right endpoint,

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to determine if $x \in A_i$ for a given $i \in \mathbb{N}$ and $x \in [0,1]$, we take $k = \lceil x * M_i \rceil$, and determine if the k^{th} term in G_i corresponds to a or n. Note that if $x = p/q$ for $p, q \in \mathbb{N}$ and $q \mid M_i$, then x will be an endpoint of a closed interval , and will be in our set wether or not k corresponds to a or n.

Let F_j and L_j denote the j^{th} Fibonacci and Lucas numbers respectively, and $\varphi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio. In this paper we introduce techniques for the analysis of G, and using these methods we prove the following theorem:

Theorem 1. *The following sets of real numbers are subsets of G:*

1. $\{\frac{F_j}{L_{j+1}},\ 1 - \frac{F_j}{L_{j+1}}\ :\ j \in \mathbb{N}\}$
2. $\{\frac{F_{2j}}{\varphi^{2t} L_{2j+1}}, 1 - \frac{F_{2j}}{\varphi^{2t} L_{2j+1}}\ :\ j, t \in \mathbb{N}$
3. $\{\frac{\sqrt{5}}{5\varphi^{2t-1}}\ :\ t \in \mathbb{N}\}$

In the course of proving the above theorem, we also prove the following identities relating Fibonacci and Lucas Numbers:

Theorem 2. 1. $\lceil \frac{F_{2k}}{L_{2k+1}} F_{2(n+(2k+1))} \rceil = \lceil \frac{F_{2k}}{L_{2k+1}} F_{2n} \rceil + \sum_{i=0}^{k-1} F_{2n+4i+3}$

2. $\lceil \frac{F_{2k-1}}{L_{2k}} F_{2(n+(4k))} \rceil = \lceil \frac{F_{2k-1}}{L_{2k}} F_{2n} \rceil + \sum_{i=0}^{k-1} F_{2n+4i+3} + \sum_{i=0}^{k-1} F_{2n+2k+4i+1}$

There are still several questions pertaining to this set that remain unanswered. In particular, do these points exhaust the Golden Ana Set, or are there more? Also, one might study the limit infimum or the limit supremum of the sequence A_n. In unpublished work, Mark McClure has demonstrated that the limit supremum has full measure. Sets constructed by this method provide an interesting new way to geometrically interpret automatic sequences (that is, sequences defined by a replacement rule), and perhaps there is a generalization of the techniques in this paper to other sequences.

References

[1] Joseph Pe, *Ana's Golden Fractal*, Fractals **11**(2003), 309–313.

[2] Clifford Pickover, *Wonders of Numbers*, Oxford University Press, 2001.