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ON RECOVERY OF A FUNCTION FROM ITS TRIGONOMETRIC INTEGRAL.

We start with a short review of results in the theory of uniqueness for trigonometric series, thus making a link between our results for trigonometric integral and classical theory.

One of the principle questions concerning trigonometric series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

which attracted attention of many mathematicians was the question of recovering the coefficients of every convergent trigonometric series from its sum. The first answer to this question was the Du Bois Reymond–Lebesgue theorem [1, 2] which states that each series everywhere convergent to a bounded Lebesgue integrable function f is the Fourier–Lebesgue series of f, i.e., its coefficients are recovered by means of Fourier formulas. This result was later generalized by Ch. Vallée-Poussin [3] on the whole class of Lebesgue integrable functions.

It was noted by A. Denjoy that not each everywhere convergent trigonometric series is a Fourier–Lebesgue series. For example, the series:

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}$$

is convergent everywhere as a sine-series with monotonically decreasing coefficients but it is not a Fourier–Lebesgue series. Thus, to recover coefficients of *each* everywhere convergent trigonometric series, one needs a more general integration process than the Lebesgue one.

A. Denjoy was the first who constructed such integration process. In 1912 he introduced an integral [4], called totalization T_{2s}, which allows to recover a function from its second symmetric Riemann derivative. This property together with the Riemann theory for trigonometric series shows that the Denjoy integral handles the problem of recovery. Other second order integrals where constructed by J. Burkill [5] (SCP-integral), J. Marcinkiewicz, A. Zygmund [6] ($T(P)$-integral) and R. James [7] (P^2-integral).

In 1989 D. Preiss and B.S. Thomson [8] introduced an integral of the first order which also solves the problem of recovery. This integral, called approximate symmetric Henstock–Kurzweil integral, recovers a measurable function from its approximate symmetric derivative and handles the problem of recovery with the aid of the Lebesgue theory of trigonometric series.

Let $\mathcal{I}$ be the set of all nondegenerate closed intervals on $\mathbb{R}$.

Definition 1. A set $\beta \subset \mathcal{I} \times \mathbb{R}$ is called *measurable approximate symmetric element*, if

(1) for each pair $(I, x) \in \beta$ the interval I is symmetric with respect to x; i.e., $I = [x-t, x+t]$; and $T = \{(x,t) : ([x-t, x+t], x) \in \beta\}$ is a measurable set on $\mathbb{R} \times (0, \infty)$,

(2) for all $x \in \mathbb{R}$: $\lim\limits_{h \searrow 0} \mu\big(\{t \in (0,h) \colon (x,t) \notin T\}\big)/h = 0.$

A finite collection of pairs $\pi = \{(I, x)\}$ is called a *division* (*a partition of interval* $[a,b]$) if for any different pairs (I_1, x_1), (I_2, x_2) from π intervals I_1 and I_2 do not overlap (and $\bigcup I = [a,b]$).

Theorem (covering theorem, [8])**.** For any measurable approximate symmetric element β there exists a set $B \subset \mathbb{R}$ of full measure such that for any closed interval with endpoints in B there is a tagged partition $\pi \subset \beta$ of this interval.

Let us denote by $\mathcal{A}_B$ the set of all measurable approximate symmetric elements containing at least one partition of every closed interval with endpoints in a set B.

In view of covering theorem the following notion is well defined.

Definition 2. A function f, defined everywhere on $\mathbb{R}$, is called *integrable in the sense of approximate symmetric Henstock–Kurzweil integral* (ASH-*integrable*), if there exists a set B of full measure and a function F on B such that for any $\varepsilon > 0$ there is an element $\beta \in \mathcal{A}_B$ such that for any division $\pi = \{([y_i, z_i], (y_i + z_i)/2)\} \subset \beta$ on the line, with $y_i, z_i \in B$, the inequality

$$\Big|\sum_{\pi}\big(f((y_i + z_i)/2)(z_i - y_i) - (F(z_i) - F(y_i))\big)\Big| < \varepsilon$$

holds. For each pair $a, b \in B$ the number $F(b) - F(a)$ is called the *approximate symmetric Henstock–Kurzweil integral* (ASH-*integral*) of the function f and is denoted as $(ASH)\int\limits_a^b f(x)dx$.

The function F, defined on B, is called an *indefinite ASH-integral* of the function f.

Definition 3. A function F defined on a measurable set E is called *approximately symmetrically continuous at* x, if

$$\operatorname{ap-lim}_{h\searrow 0}(F(x+h)-F(x-h))=0.$$

A function F defined on a measurable set E is called *approximately symmetrically differentiable at* x, if there exists a finite limit

$$\operatorname{ap-lim}_{h\searrow 0}\frac{F(x+h)-F(x-h)}{2h}=F'_{\mathrm{sap}}(x).$$

As we have already mentioned, the main property of ASH-integral is expressed in the following theorem.

Theorem (Preiss–Thomson, [8])**.** If a function F is measurable, approximately symmetrically continuous at each point of the line and has nearly everywhere approximate symmetric derivative f, then the function $f=F'_{\mathrm{sap}}$ is ASH-integrable with F being an indefinite integral.

One can verify that this property together with the Lebesgue theory for trigonometric series enables ASH-integral to handle the problem of recovery.

Theorem (Preiss–Thomson, [8])**.** If trigonometric series converges nearly everywhere to a finite function f, then functions $f(x)$ and $f(x)e^{-inx}$, $n\in\mathbb{Z}$, are ASH-integrable and

$$c_n=\frac{1}{2\pi}\int\limits_{p}^{p+2\pi}f(x)\,e^{-inx}dx$$

for almost all p.

Fourier series in practice serve as a model of periodical processes. The need to investigate also nonperiodical processes led to the generalization of series to integrals, what brought in view the so-called Fourier integral

$$\mathfrak{F}[f](t)=\int\limits_{-\infty}^{\infty}e^{itx}\hat{f}[x]\,dx,\quad \hat{f}[x]=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}f(\lambda)e^{-i\lambda x}\,d\lambda,$$

and the theory of *trigonometric integrals*; i.e., integrals of the type

$$\int\limits_{-\infty}^{\infty}e^{i\lambda x}c(\lambda)d\lambda=\lim_{\omega\to\infty}(L)\int\limits_{-\omega}^{\omega}e^{i\lambda x}c(\lambda)d\lambda.$$

The problem of recovery was naturally brought into the theory of trigonometric integrals. In the beginning of the 20th century there appeared analogs of the Vallée-Poussin theorem. One the most general was established by A. Offord [9].

Theorem (Offord)**.** Let c be locally Lebesgue integrable and the integral

$$\int_{-\infty}^{\infty} e^{i\lambda x} c(\lambda) d\lambda$$

converge everywhere to a finite locally Lebesgue integrable function f. Then

$$c(\lambda) = (C,1)\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx = \lim_{\omega\to\infty}\frac{1}{2\pi\omega}\int_{0}^{\omega} dt \int_{-t}^{t} f(x)e^{-i\lambda x}dx$$

for almost all λ.

This theorem is no longer true without the assumption f is locally Lebesgue integrable. Thus, again we need a more general process of integration to solve the problem entirely. We would show that ASH-integral handles this problem.

At first we develop the Lebesgue theory for trigonometric integrals. Let c fulfil the *condition* $\mathcal{N}_0$:

$$\lim_{u\to\pm\infty}\left\{\max_{0\leqslant h\leqslant 1}\left|\int_{u}^{u+h} c(\lambda)d\lambda\right|\right\} = 0.$$

Note, that this condition necessarily holds for any c with trigonometric integral convergent on a set of positive measure.

Integrating formally an integral $\int_{-\infty}^{\infty} e^{i\lambda x} c(\lambda)\,d\lambda$ define the function

$$L(x) = \int_{|\lambda|<1} \frac{e^{i\lambda x}-1}{i\lambda}\, c(\lambda)\,d\lambda + \int_{|\lambda|\geqslant 1} \frac{e^{i\lambda x}}{i\lambda}\, c(\lambda)\,d\lambda.$$

Theorem 1. Assume c fulfils condition $\mathcal{N}_0$. Then the function L is finite almost everywhere, approximately symmetrically continuous everywhere and approximately continuous at each point where L is finite. Moreover, if the trigonometric integral converges at x to s then there exists $L'_{\mathrm{sap}}(x) = s$.

One of the peculiarities of the passage from trigonometric series to trigonometric integrals is that to prove the analog of the Preiss–Thomson theorem, we need to use properties of the ASH-integral together with not only the Lebesgue theory, but also the Riemann theory for trigonometric integrals and so-called equiconvergence theorems. With the aid of the latter two, one can prove the following statements.

Statement 1. If a function c is locally Lebesgue integrable and fulfils condition $\mathcal{N}_0$, then for almost all μ

$$\lim_{\omega\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} c(\mu+t)\, \frac{2\sin^2 \frac{\omega t}{2}}{\omega t^2}\, dt = c(\mu).$$

Statement 2. If the trigonometric integral of a function c converges nearly everywhere, then for almost all μ

$$\lim_{\omega\to\infty} \frac{1}{\pi\omega} \int_0^{\omega} \int_{-\infty}^{\infty} c(\lambda+\mu)\, \frac{\sin\lambda t}{\lambda}\, d\lambda\, dt = c(\mu).$$

By means of the above results, an analog of the Preiss–Thomson theorem for trigonometric integrals can be proved.

Theorem 2. If a function c is locally Lebesgue integrable and its trigonometric integral converges nearly everywhere to a finite function $f(x)$, then $f(x)$ and $f(x)e^{-i\mu x}$ are ASH-integrable and for almost all μ

$$c(\mu) = (C,1)\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-i\mu x}\, dx =$$

$$= \lim_{\omega\to\infty} \frac{1}{2\pi\omega} \int_0^{\omega} \left(\int_{-t}^{t} f(x)e^{-i\mu x}\, dx \right) dt, \quad (\star)$$

where the integral over x is understood in the ASH sense and the integral over t is understood in Lebesgue sense.

Corollary 1. Every countable set is a set of uniqueness for trigonometric integral.

Corollary 2. If the trigonometric integral of a function c converges nearly everywhere to a finite function f and B is the set of points at which an indefinite ASH-integral of f is finite, then for all $a, b \in B$

$$(ASH)\int\limits_a^b f(x)\,dx = (L)\int\limits_{-\infty}^{\infty} \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}\, c(\lambda\,)d\lambda.$$

Corollary 3. If the trigonometric integral of a function c converges nearly everywhere to a finite function $f \geqslant g$ and g is locally Lebesgue (Henstock–Kurzweil) integrable, then f is also locally Lebesgue (Henstock–Kurzweil) integrable and for almost all μ the function c is recovered by the formula $(\star)$, where the integral over x is understood in the sense of Lebesgue (Henstock–Kurzweil).

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