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CONVOLUTIONS WITH THE CONTINUOUS PRIMITIVE INTEGRAL

1 Introduction.

The convolution of two functions f and g on the real line is $f*g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$. In this talk we consider convolutions for the continuous primitive integral. This integral extends the Lebesgue, Henstock–Kurzweil and wide Denjoy integrals on the real line and has a very simple definition in terms of distributional derivatives. All of the results presented are in the paper [4].

The *test functions* are $\mathcal{D} = C_c^\infty(\mathbb{R})$. The *distributions* are denoted $\mathcal{D}'$ and are the continuous linear functions on $\mathcal{D}$. For $T \in \mathcal{D}'$ and $\phi \in \mathcal{D}$ we write $\langle T, \phi\rangle \in \mathbb{R}$. If $f \in L^1_{loc}$ then $\langle T_f, \phi\rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$ defines a distribution $T_f \in \mathcal{D}'$. All distributions have derivatives of all orders that are themselves distributions. For $T \in \mathcal{D}'$ and $\phi \in \mathcal{D}$ the *distributional derivative* of T is T' where $\langle T', \phi\rangle = -\langle T, \phi'\rangle$. See [1] and [2] for more on distributions.

The following Banach space will be of importance; $\mathcal{B}_c = \{F : \mathbb{R} \to \mathbb{R} \mid F \in C^0(\overline{\mathbb{R}}), F(-\infty) = 0\}$. We use the notation $F(-\infty) = \lim_{x\to-\infty} F(x)$ and $F(\infty) = \lim_{x\to\infty} F(x)$. The extended real line is denoted $\overline{\mathbb{R}} = [-\infty, \infty]$. The space $\mathcal{B}_c$ is a Banach space under the uniform norm; $\|F\|_\infty = \sup_{x\in\mathbb{R}} |F(x)| = \max_{x\in\overline{\mathbb{R}}} |F(x)|$ for $F \in \mathcal{B}_c$. The *continuous primitive integral* is defined by taking $\mathcal{B}_c$ as the space of primitives. The space of integrable distributions is $\mathcal{A}_c = \{f \in \mathcal{D}' \mid f = F' \text{ for } F \in \mathcal{B}_c\}$. If $f \in \mathcal{A}_c$ with primitive $F \in \mathcal{B}_c$ then $\int_a^b f = F(b) - F(a)$ for $a, b \in \overline{\mathbb{R}}$. The distributional differential equation $T' = 0$ has only constant solutions so the primitive $F \in \mathcal{B}_c$ satisfying $F' = f$ is unique. Integrable distributions are then tempered and of order one. This integral, including a discussion of extensions to $\mathbb{R}^n$, is described in [3].

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It is shown there that the integration by parts formula $\int_{-\infty}^{\infty} f(x)g(x)\,dx = F(\infty)g(\infty) - \int_{-\infty}^{\infty} F(x)\,dg(x)$ holds for all $f \in \mathcal{A}_c$ with primitive $F \in \mathcal{B}_c$ and all functions $g \in \mathcal{BV}$ (bounded variation). Hence, the convolution $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$ is then well defined for all such f and g.

Theorem 1. *Let $f \in \mathcal{A}_c$ and let $g \in \mathcal{BV}$. Then (a) $f*g$ exists on $\mathbb{R}$ (b) $f*g = g*f$ (c) $\|f*g\|_\infty \leq |\int_{-\infty}^{\infty} f| \inf_{\mathbb{R}} |g| + \|f\| Vg \leq \|f\| \|g\|_{\mathcal{BV}}$ (d) $f*g \in C^0(\overline{\mathbb{R}})$, $\lim_{x\to\pm\infty} f*g(x) = g(\pm\infty)\int_{-\infty}^{\infty} f$. (e) If $h \in L^1$ then $f*(g*h) = (f*g)*h \in C^0(\overline{\mathbb{R}})$. (f) Let $x, z \in \mathbb{R}$. Then $\tau_z(f*g)(x) = (\tau_z f)*g(x) = (f*\tau_z g)(x)$. (g) For each $f \in \mathcal{A}_c$ define $\Phi_f : \mathcal{BV} \to C^0(\overline{\mathbb{R}})$ by $\Phi_f[g] = f*g$. Then Φ_f is a bounded linear operator and $\|\Phi_f\| = \|f\|$. For each $g \in \mathcal{BV}$ define $\Psi_g : \mathcal{A}_c \to C^0(\overline{\mathbb{R}})$ by $\Psi_g[f] = f*g$. Then Ψ_g is a bounded linear operator and $\|\Psi_g\| = \|g\|_{\mathcal{BV}}$. (h)* $\mathrm{supp}(f*g) \subset \mathrm{cl}(\mathrm{supp}(f) + \mathrm{supp}(g))$.

This parallels similar results when $f \in L^1$ and $g \in L^\infty$. See [1]. It is then seen that distributions in $\mathcal{A}_c$ behave substantially like L^1 functions, with the Alexiewicz norm replacing the L^1 norm.

The convolution $f*g$ exists as a Lebesgue integral almost everywhere when $f, g \in L^1$. For $f \in \mathcal{A}_c$ and $g \in L^1$ we can define the integral using a limiting procedure.

Definition 1. Let $f \in \mathcal{A}_c$ and let $g \in L^1$. Let $\{g_n\} \subset \mathcal{BV} \cap L^1$ such that $\|g_n - g\|_1 \to 0$. Define $f*g$ as the unique element in $\mathcal{A}_c$ such that $\|f*g_n - f*g\| \to 0$.

We then have the following results.

Theorem 2. *Let $f \in \mathcal{A}_c$ and $g \in L^1$. Define $f*g$ as in Definition 1. Then (a) $\|f*g\| \leq \|f\|\|g\|_1$. (b) Let $h \in L^1$. Then $(f*g)*h = f*(g*h) \in \mathcal{A}_c$. (c) For each $z \in \mathbb{R}$, $\tau_z(f*g) = (\tau_z f)*g = (f*\tau_z g)$. (d) For each $f \in \mathcal{A}_c$ define $\Phi_f : L^1 \to \mathcal{A}_c$ by $\Phi_f[g] = f*g$. Then Φ_f is a bounded linear operator and $\|\Phi_f\| = \|f\|$. For each $g \in L^1$ define $\Psi_g : \mathcal{A}_c \to \mathcal{A}_c$ by $\Psi_g[f] = f*g$. Then Ψ_g is a bounded linear operator and $\|\Psi_g\| = \|g\|_1$. (e) Define $g_t(x) = g(x/t)/t$ for $t > 0$. Let $a = \int_{-\infty}^{\infty} g_t(x)\,dx = \int_{-\infty}^{\infty} g$. Then $\|f*g_t - af\| \to 0$ as $t \to 0$. (f)* $\mathrm{supp}(f*g) \subset \mathrm{cl}(\mathrm{supp}(f) + \mathrm{supp}(g))$.

Other results, such as differentiation and integration of convolutions, are proved in [4].

A generalization of the continuous primitive integral is the *regulated primitive integral*. This is defined using regulated functions as primitives. A function is regulated if it has a left limit and a right limit at each point. The space of distributions integrable in this sense includes $\mathcal{A}_c$ and all finite, signed Radon measures. Most of the results in this talk extend to this integral. See [5].

References

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