Real Analysis Exchange
Summer Symposium 2010, pp. 14–18

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ON CONTINUITY AND COMPACTNESS OF SOME VECTOR-VALUED INTEGRALS

In this talk we elucidate some relationships between continuity and compactness properties for several extensions of the Pettis integral. The Pettis integral is the widest among the classical integrals of vector-valued functions. It is important to point out that an indefinite Pettis integral is an *absolutely continuous* set function and, in the case where the measure space is perfect, has *relatively norm compact* range.

From the descriptive point of view, the Pettis integral has several further extensions such as the McShane-Pettis integral, the Henstock-Kurzweil-Pettis integral, and Denjoy-Pettis integral (the MP, HKP, and DP integrals, in short). As a result of their generality, neither of those integrals inherits the absolute continuity and the compactness properties of the Pettis integral. Moreover, the corresponding indefinite integrals may even fail to be continuous.

First of all, we set our notation. For the most part, our notation is standard. $[a, b]$ will denote a fixed nondegenerate interval of the real line and I its closed nondegenerate subinterval. X denotes a real Banach space and X^* its dual. Given $\Phi : [a, b] \to X$, $\Delta\Phi(I)$ denotes the *increment* of Φ on I. If E is a subset of the real line, then χ_E and $\lambda(E)$ will denote the *characteristic function* of E and the *Lebesgue measure* of E. c_0 will denote, as usual, the Banach space of infinite real sequences $x = (\xi_1, \xi_2, \dots)$ such that $\xi_n \to 0$ as $n \to \infty$ with the norm defined by $\|x\|_\infty = \sup_n |\xi_n|$.

Now, we summarize some notions related to the integration and differentiation of vector-valued functions. Let $\Phi : [a, b] \to X$ and let $E \subset [a, b]$.

Definition 1. Let $f : E \to X$. f is a *scalar derivative* (resp. an *approximate scalar derivative*) of Φ on E if for each x^* in X^* the function $x^*\Phi$ is differ-

Mathematical Reviews subject classification: Primary: 26A39, 28B05; Secondary: 46G10
Key words: Henstock-Kurzweil-Pettis integral, Denjoy-Pettis integral, McShane-Pettis integral, Schur property

entiable (resp. approximately differentiable) to x^*f almost everywhere on E (the exceptional set may vary with x^*).

Definition 2. Φ is said to be AC (resp. AC_*) on E if for each positive number ε there exists a positive number η such that

$$\left\| \sum_{k=1}^{K} \Delta\Phi(I_k) \right\| < \varepsilon$$

for each finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^{K}$ that have both endpoints in E (resp. have at least one endpoint in E) and satisfy

$$\lambda\left(\bigcup_{k=1}^{K} I_k \right) < \eta.$$

The Pettis integral admits two equivalent definitions. We begin with the classical [4]. Let $f : [a,b] \to X$.

Definition 3. The function f is *Pettis integrable* on $[a,b]$ if for each Lebesgue measurable set E in $[a,b]$ there is a vector $\nu_f(E)$, the Pettis integral of f on E, in X such that the Lebesgue integral $\int_E x^* \circ f \, d\lambda$ exists and is equal to $x^*(\nu_f(E))$ for all x^* in X^*.

The descriptive definition [3] of the Pettis integral reads as follows.

Definition 4. The function f is Pettis integrable on $[a,b]$ if there is a function $\Phi : [a,b] \to X$ such that $\Phi(a) = 0$, Φ is AC on $[a,b]$, and f is a scalar derivative of Φ on $[a,b]$.

The MP integral is obtained by relaxing this definition in an obvious way.

Definition 5. The function f is *MP integrable* on $[a,b]$ if there exists a function $\Phi : [a,b] \to X$ such that $\Phi(a) = 0$, for each x^* in X^* the function $x^*\Phi$ is AC on $[a,b]$, and f is a scalar derivative of Φ on $[a,b]$.

Next, we define two Denjoy type extensions of the MP integral.

Definition 6 (ACG and ACG_* Functions)**.** Let $\Phi : [a,b] \to X$ and let $E \subset [a,b]$. Φ is said to be ACG (resp. ACG_*) on E if E can be written as a countable union of sets on each of which Φ is AC (resp. AC_*).

Definition 7. A function $f : [a,b] \to X$ is *HKP integrable* (resp. *DP integrable*) on $[a,b]$ if there exists a function $\Phi : [a,b] \to X$ such that $\Phi(a) = 0$, for each x^* in X^* the function $x^*\Phi$ is ACG_* (resp. ACG and continuous) on $[a,b]$, and f is a scalar derivative (resp. an approximate scalar derivative) of Φ on $[a,b]$.

A straightforward argument can be given to show that the function Φ in the definition of the DP integral is unique. Such a function will be referred to as the *indefinite* integral of the function f. Given I, we write

$$\int_I f = \Delta\Phi(I).$$

We provide two examples to better illustrate the distinction between the Pettis integral and the MP integral. The first [2], [1], [3] shows that an indefinite MP integral is not necessarily continuous and does not necessarily has relatively norm compact range. Let $\{I_n\}_{n=1}^{\infty}$ be a fixed sequence of intervals in $[a,b]$ such that $b_n = \max I_n < \min I_{n+1}$ for each n and $b_n \to b$ as $n \to \infty$.

Example 1. *Let $\{e_n\}_{n=1}^{\infty}$ be the standard unit vector basis of c_0. Define a function $f : [a,b] \to c_0$ by*

$$f(t) = \sum_n \left(\frac{\chi_{I_{2n-1}}(t)}{2\lambda(I_{2n-1})} - \frac{\chi_{I_{2n}}(t)}{2\lambda(I_{2n})} \right) e_n.$$

Then f is MP integrable on $[a,b]$, the indefinite integral of f is not *continuous at b and the set $\left\{\int_{I_{2n-1}} f\right\}_{n=1}^{\infty} = \left\{\frac{e_1}{2}, \frac{e_2}{2}, \dots\right\}$ is* not *relatively norm compact.*

The second [3], which is a refinement of the first, shows that a continuous indefinite MP integral is not necessarily absolutely continuous.

Example 2. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in c_0 defined by*

$$e_1, \underbrace{\frac{e_2}{2}, \frac{e_2}{2}}_{2}, \underbrace{\frac{e_3}{3}, \frac{e_3}{3}, \frac{e_3}{3}}_{3}, \dots, \underbrace{\frac{e_k}{k}, \dots, \frac{e_k}{k}}_{k}, \dots.$$

Define a function $g : [a,b] \to c_0$ by

$$g(t) = \sum_n \left(\frac{\chi_{I_{2n-1}}(t)}{2\lambda(I_{2n-1})} - \frac{\chi_{I_{2n}}(t)}{2\lambda(I_{2n})} \right) x_n.$$

Then g is MP integrable on $[a,b]$, the indefinite integral of g is continuous on $[a,b]$ and g is not *Pettis integrable on $[a,b]$.*

We begin formulating our positive results with a necessary and sufficient condition in order that an indefinite DP integral have relatively norm compact range.

Theorem 1. *Suppose $\Phi : [a,b] \to X$ is an indefinite DP integral. Then the following two statements are equivalent:*

- *Φ is continuous on $[a,b]$.*
- *The set $\{\Delta\Phi(I) : I \subset [a,b]\}$ is relatively norm compact.*

Recall that a real Banach space X is said to have the *Schur property* (or to be a *Schur space*, in short) if each weakly null sequence in X converges in norm. The next theorem gives a characterization of Schur spaces in terms of the continuity and the compactness properties of vector-valued integrals.

Theorem 2. *The following three statements are equivalent:*

- *X is a Schur space.*
- *Each indefinite HKP (resp. DP) integral $\Phi : [a,b] \to X$ is continuous on $[a,b]$.*
- *The set $\{\Delta\Phi(I) : I \subset [a,b]\}$ is relatively norm compact for each indefinite HKP (resp. DP) integral $\Phi : [a,b] \to X$.*

In the case of the DP integral this characterization can be refined in the following way.

Theorem 3. *The following two statements are equivalent:*

- *X is a Schur space.*
- *Each indefinite DP integral $\Phi : [a,b] \to X$ is continuous save at most on a countable set.*

On the other hand, the more general class of Banach spaces containing no isomorphic copy of c_0 can be characterized in terms of similar continuity and compactness properties of indefinite HKP and MP integrals.

Theorem 4. *The following six statements are equivalent:*

- *X contains no isomorphic copy of c_0.*
- *Each indefinite HKP integral $\Phi : [a,b] \to X$ is continuous save at most on a countable set.*
- *Each indefinite MP integral $\Phi : [a,b] \to X$ is continuous save at most on a countable set.*
- *Each indefinite MP integral $\Phi : [a,b] \to X$ is continuous on $[a,b]$.*
- *Each indefinite MP integral $\Phi : [a,b] \to X$ is AC on $[a,b]$.*
- *The set $\{\Delta\Phi(I) : I \subset [a,b]\}$ is relatively norm compact for each indefinite MP integral $\Phi : [a,b] \to X$.*

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