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IRREGULAR RECURRENCE IN COMPACT METRIC SPACES

1 Introduction.

Let (X, d) be a compact metric space, and $\mathcal{C}(X)$ the set of continuous maps $f : X \to X$. By $\omega(x, f)$ we denote the *ω-limit set* of x which is the set of limit points of the *trajectory* $\{f^i(x)\}_{i\geq 0}$ of x, where f^i denotes the ith iterate of f. We consider sets $W(f)$ of *weakly almost periodic points* of f, and $QW(f)$ of *quasi-weakly almost periodic points* of f. They are defined as follows: For $x \in X$ and $t > 0$, let

$$\Psi_x(f,t) = \liminf_{n\to\infty} \tfrac{1}{n}\#\{0 \leq j < n; d(x, f^j(x)) < t\}, \tag{1}$$

$$\Psi^*_x(f,t) = \limsup_{n\to\infty} \tfrac{1}{n}\#\{0 \leq j < n; d(x, f^j(x)) < t\}. \tag{2}$$

Point $x \in W(f)$ if and only if $\Psi_x(f,t) > 0$, for every positive t.

Point $x \in QW(f)$ if and only if $\Psi^*_x(f,t) > 0$, for every positive t.

Obviously, $W(f) \subseteq QW(f)$. The properties of $W(f)$ and $QW(f)$ were studied in the nineties by Z. Zhou et al, see [8] for references. The points in $IR(f) = QW(f) \setminus W(f)$ are *irregularly recurrent points*, i.e. points x such that $\Psi^*_x(f,t) > 0$ for any $t > 0$, and $\Psi_x(f,t_0) = 0$ for *some* $t_0 > 0$.

Denote by $R(f)$ the set of *recurrent points*, and by $UR(f)$ the set of *uniformly recurrent points* of f. Thus, $x \in R(f)$ if, for every neighborhood U of x, $f^j(x) \in U$ for infinitely many $j \in \mathbb{N}$, and $x \in UR(f)$ if, for every neighborhood U of x there is a $K > 0$ such that every interval $[n, n + K]$,

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$n \in \mathbb{N}$, contains a $j \in \mathbb{N}$ with $f^j(x) \in U$. Recall that $x \in R(f)$ if and only if $x \in \omega(x, f)$, and $x \in UR(f)$ if and only if $\omega(x, f)$ is a *minimal set*, i.e., a closed set $\emptyset \neq M \subseteq X$ such that $f(M) = M$ and no proper subset of M hasthis property. The following relations are obvious:

$$UR(f) \subseteq W(f) \subseteq QW(f) \subseteq R(F).$$

2 Relations with topological entropy.

Let (Σ_2, σ) be the shift on the set Σ_2 of sequences of two symbols, $0, 1$, equipped with a metric ρ of pointwise convergence, say, $\rho(\{x_i\}_{i\geq 1}, \{y_i\}_{i\geq 1}) = 1/k$ where $k = \min\{i \geq 1; x_i \neq y_i\}$.

LEMMA 1. *$IR(\sigma)$ is non-empty, and contains a transitive point.*

PROOF. Let

$$k_{1,1}, k_{1,2}, k_{2,1}, k_{2,2}, k_{2,3}, k_{3,1}, \cdots, k_{3,4}, k_{4,1}, \cdots, k_{4,5}, k_{5,1}, \cdots$$

be an increasing sequence of positive integers. Let $\{B_n\}_{n\geq 1}$ be a sequence of all finite blocks of digits 0 and 1. Put $A_0 = 10$, $A_1 = (A_0)^{k_{1,1}} 0^{k_{1,2}} B_1$ and, in general,

$$A_{n+1} = A_n (A_0)^{k_{n+1,1}} (A_1)^{k_{n+1,2}} \cdots (A_n)^{k_{n+1,n+1}} 0^{k_{n+1,n+2}} B_{n+1}. \tag{3}$$

Denote by $|A|$ the length of a finite block of 0's and 1's, and let

$$a_n = |A_n|, \; b_n = |B_n|, \; c_n = a_n - b_n - k_{n,n+1}, \; n \in \mathbb{N}, \tag{4}$$

and

$$\lambda_{n,m} = \left| A_n (A_0)^{k_{n+1,1}} (A_1)^{k_{n+1,2}} \cdots (A_m)^{k_{n+1,m+1}} \right|, \; m, n \in \mathbb{N}, m \leq n+1. \tag{5}$$

By induction we can take the numbers $k_{i,j}$ such that

$$k_{n,m+1} = n \cdot \lambda_{n,m}, \; m, n \in \mathbb{N}, m \leq n+1. \tag{6}$$

Let $N(A)$ be the cylinder of all $x \in \Sigma_2$ beginning with a finite block A. Then every $N(B_n)$ is an open set in Σ_2, and $N(B_n), n \geq 0$, is a base of the topology of Σ_2. Clearly, $\bigcap_{n=1}^{\infty} N(A_n)$ contains exactly one point; denote it by u.

Since $\sigma^{a_n - b_n}(u) \in N(B_n)$, i.e., since the trajectory of u visits every $N(B_n)$, u is a transitive point of σ. Moreover, $\rho(u, \sigma^j(u)) = 1$, whenever $c_n \leq j < a_n - b_n$. By (6) it follows that $\Psi_u(\sigma, t) = 0$ for every $t \in (0, 1)$. Consequently, $u \notin W(\sigma)$.

It remains to show that $u \in QW(\sigma)$. Let $t \in (0,1)$. Fix an $n_0 \in \mathbb{N}$ such that $1/a_{n_0} < t$. Then, by (3),

$$\# \left\{ j < \lambda_{n,n_0}; \rho(u, \sigma^j(u)) < t \right\} \geq k_{n,n_0}, \ n \geq n_0 - 1,$$

hence, by (5) and (6),

$$\lim_{n\to\infty} \frac{\# \left\{ j < \lambda_{n,n_0}; \rho(u, \sigma^j(u)) < t \right\}}{\lambda_{n,n_0}} \geq \lim_{n\to\infty} \frac{k_{n+1,n_0}}{\lambda_{n,n_0}}$$
$$= \lim_{n\to\infty} \frac{k_{n+1,n_0}}{\lambda_{n,n_0-1} + a_{n_0} k_{n+1,n_0}} = \lim_{n\to\infty} \frac{n}{1 + a_{n_0} n} = \frac{1}{a_{n_0}}.$$

Thus, $\Psi^*_u(\sigma, t) = 1/a_{n_0}$ and, by the definition of set $QW(\sigma)$, $u \in QW(\sigma)$. □

LEMMA 2. *Let f be a continuous map of the interval with positive topological entropy. Then $IR(f) \neq \emptyset$.*

PROOF. When $h(f) > 0$, then f^n is strictly turbulent for some n (there exist two disjoint compact intervals K_0, K_1 and a positive integer m, such that $f^m(K_0) \cap f^m(K_1) \supset K_0 \cup K_1$, see [1], Theorem IX, 28). This condition is equivalent to existence of a continuous map $g : X \subset I \to \Sigma_2$, where X is of Cantor type, such that $g \circ f^n(x) = \sigma \circ g(x)$ for every $x \in X$, and such that each point in Σ_2 is the image of at most two points in X ([1], Proposition II, 15). From Lemma 1, there is a $u \in IR(\sigma)$. Hence, for every $t > 0$, $\Psi^*_u(\sigma, t) > 0$, and there is an $s > 0$ such that $\Psi_u(\sigma, s) = 0$. There are at most two preimages, u_0 and u_1, of u. Then, by the continuity, $\Psi_{u_i}(f^n, r) = 0$, for some $r > 0$ and $i = 0, 1$, and $\Psi^*_{u_i}(f^n, k) > 0$. for at least one $i \in \{0, 1\}$ and every $k > 0$. Thus, $u_0 \in IR(f^m)$ or $u_1 \in IR(f^m)$ and, by the fact that $IR(f) = IR(f^m)$ for every integer m and every $f \in \mathcal{C}(X)$, $IR(f) \neq \emptyset$. □

LEMMA 3. *For a continuous map f of the interval I with zero topological entropy, the set $IR(f)$ is empty.*

PROOF. Assume $x \in IR(f)$. Point x is recurrent, so it belongs to the set $\omega(f) = \bigcup_{x \in X} \omega_f(x)$. Since x cannot be periodic, it belongs to an infinite $\omega_f(y)$. Since $h(f) = 0$, we have $\omega_f(y) = Q \cup S$, where Q is a minimal set of Cantor type, and S a countable set of isolated points such that $S \cap R(f) = \emptyset$([2], Theorem 6.2). It follows that x belongs to the minimal set Q. Consequently, x is uniformly recurrent, contrary to $UR(f) \cap IR(f) = \emptyset$. □

Moreover, it can be proved that if $f \in \mathcal{C}(X)$ and $R(f)$ denotes the set of recurrent points of f, then $f^{-1}(QW(f)) \cap R(f) \subseteq QW(f)$ and $f^{-1}(W(f)) \cap$

$R(f) \subseteq W(f)$.

From this fact together with lemmas 1, 2 and 3, we have the following result:

THEOREM 1. *For a continuous map f of the interval, the conditions $h(f) > 0$ and $IR(f) \neq \emptyset$ are equivalent.*

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