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FRACTIONAL ITERATION OF SERIES AND TRANSSERIES

Definition 1. Given functions $T\colon X \to X$ and $\Phi\colon \mathbb{R} \times X \to X$, we say that Φ is a ***real iteration group*** for T iff

$$\Phi(s+t,x) = \Phi\big(s,\Phi(t,x)\big), \tag{1}$$
$$\Phi(0,x) = x, \tag{2}$$
$$\Phi(1,x) = T(x), \tag{3}$$

for all $s,t \in \mathbb{R}$ and $x \in X$.

Here, we investigate this in the setting of transseries. An example follows, essentially due to Calyey in 1860 [3]. Consider a series of the form

$$T(x) = x\left(1 + \sum_{j=1}^{\infty} c_j x^{-j}\right) \tag{4}$$
$$= x + \sum_{j=1}^{\infty} c_j x^{-j+1} = x + c_1 + c_2 x^{-1} + c_3 x^{-2} + \cdots$$

Such a series admits an iteration group of the same form. That is,

$$\Phi(s,x) = x\left(1 + \sum_{j=1}^{\infty} \alpha_j(s) x^{-j}\right). \tag{5}$$

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In fact, $\alpha_j(s)$ is $sc_j +$ {polynomial in $s, c_1, c_2, \ldots, c_{j-1}$ with rational coefficients, of degree $j-1$ in s}. The first few terms:

$$\Phi(s,x) = x + sc_1 + sc_2x^{-1} + \left(sc_3 + \frac{s(1-s)}{2}c_1c_2\right)x^{-2} + \left(sc_4 + \frac{s(1-s)}{2}(2c_1c_3 + c_2^2) + \frac{s(1-s)(1-2s)}{6}c_1^2c_2\right)x^{-3} + \cdots .$$

Theorem 2. *Let $T(x)$ be the power series* (4)*. Define $\alpha_j : \mathbb{R} \to \mathbb{R}$ recursively by*

$$\begin{aligned} \alpha_1(s) &= sc_1, \\ \alpha_j(s) &= s\left(c_j - \int_0^1 \sum_{j_1+j_2=j} (-j_1+1)\alpha_{j_1}(u)\alpha'_{j_2}(0)\,du\right) \\ &\quad + \int_0^s \sum_{j_1+j_2=j} (-j_1+1)\alpha_{j_1}(u)\alpha'_{j_2}(0)\,du. \end{aligned}$$

Then the series Φ defined formally by (5) *is a real iteration group for T.*

A second example is a "transseries".

$$T(x) = x\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{j,k}x^{-j}e^{-kx}\right), \qquad c_{0,0} = 1. \tag{6}$$

The set

$$B = \{\,(j,k) : k \geq 0, j \geq 0, (j,k) \neq (0,0)\,\} \tag{7}$$

is a semigroup under addition, well ordered by asymptotic comparison (as $x \to \infty$).

Theorem 3. *Let B be as in* (7)*. Then the transseries* (6) *admits a real iteration group supported by the same set $\{\, x^{1-j}e^{-kx} : j,k \geq 0\,\}$.*

Another simple example shows that real iteration group of that type need not always exist. Let $B \subseteq \mathbb{Z}^3$ be

$$B = \{\,(j,0,0) : j \geq 1\,\} \cup \{\,(j,k,0) : k \geq 1\,\} \cup \{\,(j,k,l) : l \geq 1\,\}. \tag{8}$$

Then $\left\{ x^{-j}e^{-kx}e^{-lx^2} : (j,k,l) \in B \right\}$ is a semigroup, but not well ordered.

Theorem 4. *Let B' be a well ordered subset of* (8). *The transseries $T(x) = x(1 + x^{-1} + e^{-x^2})$ admits no real iteration group of the form*

$$\Phi(s, x) = x \left(1 + \sum_{(j,k,l) \in B'} \alpha_{jkl}(s) x^{-j} e^{-kx} e^{-lx^2} \right).$$

Another approach involves ***Abel's Equation***. We write $\mathcal{P}$ for the set of (well-based) large positive transseries, and $\operatorname{expo} T$ for the exponentiality of T. For both, see [7].

Theorem 5. *Let $T \in \mathcal{P}$ with $\operatorname{expo} T = 0$. Then there is $V \in \mathcal{P}$ such that:* (i) *If $T > x$, then $V \circ T \circ V^{[-1]} = x+1$;* (ii) *If $T < x$, then $V \circ T \circ V^{[-1]} = x-1$.*

Corollary 6. *Let $T \in \mathcal{P}$ with $\operatorname{expo} T = 0$. Then there exists real iteration group $\Phi(s, x)$ for T.*

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