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PROBABLILITY MEASURES ON SHRINKING NEIGHBORHOODS

Abstract

In this paper we look at evenly distributed probability measures on tubular neighborhoods around certain sets, and consider the converged measure once the neighborhood is shrunk around the set. For fractals with the Open Set Condition, the measure converges to an evenly distributed probability measure that matches the Hausdorff dimension of the set. However, for targets where the length of the concentric circles is infinite, the measure on the shrinking neighborhood converges to a point measure located only on the center of the target. We will consider other situations where the converged measure on the shrunken neighborhood omits part of the set. We will also look at connections to Minkowski Content and other measures that take into account curvature or other characteristics of the set.

1 Introduction.

We consider distance tubes around objects, which we define now:

Definition 1.1. *We say x is in a distance tube of $\delta > 0$ of an object M in $\mathbb{R}^n$ if* $\operatorname{dist}(x, M) < \delta$ *where* $\operatorname{dist}(x, M) \equiv \inf\{\|x - m\| | m \in M\}$.

Over the years, there have been many interesting results concerning tubes around objects. For example, in 1939, Hermann Weyl gave a formula for the volume of a tube around a m-dimensional submanifold in n-dimensional space. This formula was a polynomial which took into account the thickness of the tube and the curvature of the object. For this formula to work, the tube needed to be small enough so that it could follow the curvature of the submanifold and not have any kinks or corners. Also, Weyl did not quite consider the distance tubes we are considering here, in that the tube had to come from the normal to the submanifold. As an example, consider a finite length line in $\mathbb{R}^2$. The distance tube we are considering in this paper has two "nubs" around the endpoints of the line, whereas Weyl's tubes do not. For a detailed account and proof of Weyl's formula, an excellent reference is [5].

More recently, Michel Lapidus and Erin Pearse gave formulas for the volumes of tubes around self-similar tilings and fractals, such as the Koch Curve (see [9] p.65-67). Their results can be found in [8].

For our purposes, we are not interested in the volume of such tubes, but rather where the volume collects as $\delta \to 0$ in the definition of the distance tube. That is, we set up a probability measure on a distance tube around a set M in $\mathbb{R}^n$ as follows:

Definition 1.2. *Let*

$$P_{M_\delta}(B) = \frac{\int_B \chi_{M_\delta}(x)dx}{\int_{\mathbb{R}^n} \chi_{M_\delta}(x)dx}$$

where B is a Borel set, M_δ is the δ-tube around M, and χ_{M_δ} is the characteristic function on M_δ.

This type of measure originally stemmed from the study of Gibbs measures and simulated annealing. For more information on these Gibbs measures and their applications, consult [1] or [6]. In [11] and [12], we studied the weak convergence of a sequence of Gibbs measures as $\lambda \to \infty$, where weak convergence is as follows:

Definition 1.3. *A sequence of probability measures $\{P_n\}_n$ converges weakly to a probability measure P, denoted $P_n \rightharpoonup P$, if*

$$\int \phi dP_n \to \int \phi dP$$

for all bounded continuous real-valued functions ϕ.

2 Results.

2.1 Rectifiable curves.

For rectifiable curves, it can be proved that the probability measure converges weakly to a probability distribution on the space when $\delta \to 0$. This includes spaces that have a finite number of corners and cusps, such as semialgebraic and subanalytic sets. This fact comes from a theorem in [4].

2.2 Fractals.

Fractals often have an infinite number of corners. However, we find that the probability measure defined above will distribute evenly for self-similar fractals with Hutchinson's Open Set Condition, which states that the fractal can be separated into self-similar parts by open sets, where the Hausdorff measure of the total intersection between the open sets can be bounded by any $\epsilon > 0$. See [7], [2], or [11] for more specfics. Examples of these types of fractals in $\mathbb{R}^2$ include the Koch curve and the $\frac{1}{4}$-Cantor set crossed with itself (see [10] p.32-33).

Therefore, we find:

Theorem 2.1. *For a self-similar fractal F Hausdorff dimension s (denoted $\mathcal{H}^s$), with Hutchinson's Open Set Condition, $P_{F_\delta}(B)$ converges weakly to*

$$\frac{\mathcal{H}^s \llcorner F}{\mathcal{H}^s(F)}(B)$$

as $\delta \to 0$, where $(\mathcal{H}^s \llcorner F)(B) = \mathcal{H}^s(F \cap B)$.

This result can be extended for fractals that do not have Hutchinson's Open Set Condition (overlapping or generic fractals, see [9]). A current theorem dealing with such fractals on the real line follows.

Theorem 2.2. *Let $T_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, N$, $N \geq 2$, be linear similitudes given by $T_i x = \tau_i x$ where $\tau_i \neq 0$ and $\sum_{i=1}^N |\tau_i| < 1$. Then for $\mathcal{L}^N$ almost all $(c_1, \ldots, c_N) \in \mathbb{R}^N$ the non-empty compact invariant set K,*

$$K = \bigcup_{i=1}^N (T_i + c_i)(K).$$

Then

$$P_{K_\delta}(B)$$

converges weakly to

$$\frac{\mathcal{H}^s \llcorner K}{\mathcal{H}^s(K)}(B)$$

as $\delta \to 0$.

In the future I would like to extend the above theorem to include overlapping fractals in arbitrary dimensions and more general types of fractals, such as Julia sets and random fractals. I would also like to explore more complicated targets and how the measures would distribute. Finally, I would like to look at Gibbs measures weighted by curvature, connected to the curvature measure described by Federer in [3].

2.3 Targets.

Although the probability measure distributes evenly over rectifiable curves and fractals, we can construct spaces such that the measure will not distribute evenly. To this end, we study "targets," that is, concentric circles with decreasing radii. When the total sum of the length of the circles have infinite length, the measure will concentrate on certain spots. By having multiple targets, we can distribute the measure in different ways. To that end, we arrive at the following three theorems, taken from [11]:

Theorem 2.3. *For a target T made up of concentric circles of decreasing radii a_n for $n \in \mathbb{N}$, $a_n \in \mathbb{R}$, and $\sum_{n=1}^{\infty} a_n = \infty$, $P_{T_\delta}(B)$ converges weakly to a point mass of probability 1 located at the center of T.*

Theorem 2.4. *For targets T_-, T_+ some distance apart from each other, made up of concentric circles of radii $\frac{1}{n}$ for T_- and concentric circles of radii $\frac{1}{2n}$ for T_+, for $n \in \mathbb{N}$, and $T = T_- \cup T_+$, $P_{T_\delta}(B)$ converges weakly to a point mass of probability $2/3$ located at the center of T_-, and a point mass of probability $1/3$ located at the center of T_+.*

Theorem 2.5. *For targets T_-, T_+ some distance apart from each other, made up of concentric circles of radii $\frac{2}{n}$ for T_- and concentric circles of radii $1 + \frac{1}{e^n}$ for T_+, for $n \in \mathbb{N}$, and $T = T_- \cup T_+$, $P_{T_\delta}(B)$ converges weakly to a point mass of probability $1/2$ located at the center of T_-, and a distribution of total probability $1/2$ located around the center circle of T_+.*

Therefore, by balancing out the measures of a target centered at a point and a target centered at a circle, the resulting measure divides its time between a zero dimensional object and a one dimensional object. For future work and a possible undergraduate research project, I would like to construct spaces where the measure divides its time between spaces of more dimension, for example, a point and a sphere, or a point, circle and sphere.

2.4 Future work: other measures.

For future work, I would like to focus on curvature measures, similar to those described by Federer in [3]. This measure would record infinite curvature at any corner or cusp, and therefore would focus the probability measure on those points and disappear whenever the set M was merely rounded or flat. I would also like to consider the relationship between $P_{M_\delta}(B)$ and Minkowski Content, defined in [4] as

$$\lim_{\delta \to 0^+} \frac{\mathcal{L}^n\{x : \mathrm{dist}(x, M) < \delta\}}{\alpha(n-m)\delta^{n-m}},$$

where $\mathcal{L}$ is Lebesgue measure, n is the dimension of the ambient space, m is the dimension of the set M, and $\alpha(n-m)$ is the volume of the unit $(n-m)$-dimensional sphere. One can see that this has many similarities to $P_{M_\delta}(B)$, but also some key differences. In Federer's definition, m needed to be an integer, therefore fractals such as the Cantor set could not be discussed. Also, Minkowski Content is not a probability measure, nor is it always even a measure. However, the reason why the centers of the targets in the last section end up with the entire measure is because these points have higher Minkowski Content then the points on the outside circles.

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