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Martina Zähle, Mathematical Institute, Friedrich Schiller University Jena, 07737 Jena, Germany. email: martina.zaehle@uni-jena.de

LOCAL AND GLOBAL CURVATURES OF SELF-SIMILAR FRACTALS

The results presented in the talk have been completed in the weeks after the Symposium in collaboration with Jan Rataj and have led to the joint paper [1] to which I am referring below:

In recent years first attempts to investigate a second order fractal 'differential' geometry have been made, see Winter, [2], Zähle [5], Winter and Zähle [4], Winter [3]. The main idea was to approximate fractal sets in $\mathbb{R}^d$ by small neighborhoods and to use known results from singular curvature theory in convex geometry and, more generally, geometric measure theory in the sense of Federer for these neighborhoods provided they have the desired structure. It turned out that this is the case for many self-similar sets satisfying the Open Set Condition. In order to obtain limit results for the appropriately rescaled global curvatures the renewal theorem from probability theory and asymptotic analysis has been used. (For the special case of the Minkowski content in $\mathbb{R}^1$ this goes back to Lapidus/ Pomerance and Falconer and in $\mathbb{R}^d$ to Gatzouras.) Moreover, weak limits of the corresponding curvature measures have been obtained as a consequence taking into regard the self-similarity property and Prohorov's theorem on weak compactness of tight families of measures.

In [1] we suggest another approach. We start with a result concerning the existence of local fractal curvatures at almost all points of the self-similar set F (Section 2). (The differential geometric analogue are the symmetric functions of principal curvatures of smooth submanifolds.) For this we mainly use the scaling properties of the curvature measures and Birkhoff's ergodic theorem for an associated dynamical system. The positive reach assumption on the closure of the complement of the parallel sets of F for almost all distances is the same as in the former papers, but the integrability condition is essentially weakened. However, here we get only convergence results in the sense of average limits.

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By the choice of an appropriate net of locally homogeneous neighborhoods $A_F(x, \varepsilon)$, $x \in F$, $\varepsilon < \varepsilon_0$, for the construction of these local curvatures, we can easily derive the existence of related global fractal curvatures which simplifies the proofs and extends the corresponding results from [2], [5] and [4]. The weak convergence of the associated curvature measures then follows as in [2] and [4]. Moreover, the local curvatures can now be interpreted as densities of the fractal limit measures with respect to D-dimensional Hausdorff measure on F, where D equals the Hausdorff dimension. (See Section 3.)

Finally, in Section 4 we study the examples of the Cantor dust in the plane and the Menger sponge in space which demonstrate some typical phenomena. They do not satisfy the assumptions from the former papers: in particular, the Euler number of the parallel sets $F(\varepsilon)$ of F with distance ε is unbounded at neighborhoods of certain critical values of ε. Nevertheless, they fit into the approach of our paper. (See also [3] for further conditions and examples.)

References

[1] J. Rataj, M. Zähle, *Curvature densities of self-similar sets* (Submitted), arXiv:1009.6162.

[2] S. Winter, *Curvature Measures and Fractals*, Diss. Math. **453** (2008) 1–66.

[3] S. Winter, *Curvature bounds for neighborhoods of self-similar sets*, Preprint.

[4] S. Winter, M. Zähle, *Fractal curvature measures of self-similar sets*, (Submitted), arXiv:1007.0696.

[5] M. Zähle, *Lipschitz-Killing curvatures of self-similar random fractals*, Trans. Amer. Math. Soc., to appear, arXiv:1009.6166.