Convergence of ergodic averages for many group rotations

Zoltan Buczolich

Suppose that G is a compact Abelian topological group, m is the Haar measure on G and $f : G \to \mathbb{R}$ is a measurable function. Given (n_k), a strictly monotone increasing sequence of integers we consider the nonconventional ergodic/Birkhoff averages

$$M_N^\alpha f(x) = \frac{1}{N+1} \sum_{k=0}^{N} f(x + n_k \alpha).$$

The f-rotation set is
$\Gamma_f = \{\alpha \in G : M_N^\alpha f(x)$ converges for m a.e. x as $N \to \infty.\}$ We prove that if G is a compact locally connected Abelian group and $f : G \to \mathbb{R}$ is a measurable function then from $m(\Gamma_f) > 0$ it follows that $f \in L^1(G)$. A similar result is established for ordinary Birkhoff averages if $G = Z_p$, the group of p-adic integers. However, if the dual group, $\widehat{G}$ contains infinitely many multiple torsion then such results do not hold if one considers nonconventional Birkhoff averages along ergodic sequences. What really matters in our results is the boundedness of the tail, $f(x + n_k \alpha)/k$, $k = 1, \ldots$ for a.e. x for many α, hence some of our theorems are stated by using instead of Γ_f slightly larger sets, denoted by $\Gamma_{f,b}$. This is a joint work with G. Keszthelyi.